Poisson and Diffusion Approximation of Stochastic Schrödinger Equations with Control

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Abstract

"Quantum trajectories" are solutions of stochastic differential equations. Such equations are called "Stochastic Schrödinger Equations" and describe random phenomena in continuous measurement theory of Open Quantum System. Many recent developments deal with the control theory in such model (optimization, monitoring, engineering...). In this article, stochastic models with control are mathematically and physically justified as limit of concrete discrete procedures called "Quantum Repeated Measurements". In particular, this gives a rigorous justification of the Poisson and diffusion approximation in quantum measurement theory with control. Furthermore we investigate some examples using control in quantum mechanics.

Introduction

One of the topic in Quantum Open System theory concerns the study of the evolution of a small quantum system $\mathcal{H}_0$ undergoing an indirect and continuous measurement (the small system is in contact with environment and the measurement is performed on the environment). In this context, the evolution of the system is usually described by classical stochastic differential equations called *"Stochastic Schrödinger Equations"*. Essentially, two kind of equations are considered:

1. The "diffusive equation" (Homodyne detection experiment) is given by

$$d\rho_t = L(\rho_t)dt + [\rho_t C^\star + C\rho_t - Tr\left(\rho_t(C + C^\star)\right)\rho_t]dW_t \tag{1}$$

where W_t describes a one-dimensional Brownian motion.

2. The “jump equation” (Resonance fluorescence experiment) is

$$d\rho_t = L(\rho_t)dt + \left[\frac{\mathcal{J}(\rho_t)}{Tr[\mathcal{J}(\rho_t)]} - \rho_t\right](d\tilde{N}_t - Tr[\mathcal{J}(\rho_t)]dt) \tag{2}$$

where $\tilde{N}_t$ is a counting process with stochastic intensity $\int_0^t Tr[\mathcal{J}(\rho_s)]ds$.

Solutions of such equations are called *"Quantum Trajectory"*; they describe the evolution of the state of the system perturbed by the continuous indirect measurement.

Recent progress and developments in quantum optics and quantum information theory need a highest precision in experience process using measurement [15] (sensitivity, miniaturization, optimization...). It imposed then quantum systems to be controlled. Two types of controls are usually considered in Quantum Mechanics: deterministic and stochastic control.

A laser monitoring a qubit, i.e a two level atom system, is a basic example of deterministic control application. The action of control is then characterized by the laser intensity (the term deterministic is related to the fact that we consider the intensity to be a deterministic function of time). Often, it is called *"Open Loop Control"*. Such experiences are used in order to prepare systems in specific states for quantum computing.

The notion of stochastic control is, here, directly connected with the procedure of measurement. Depending on the result of measurement, a control operation is performed in order to modify the evolution of the system. As a result of measurement is random in Quantum Mechanics (one of the axiom of the theory), the control becomes also random (it justifies the term of stochastic control). This kind of control is particularly used in engineering when some constraint of precision and optimization must be followed. Usually such control is called *"Closed Loop Control"* or *"Feedback Control"*.

From a theoretical point of view, an important question is to lay out a mathematical setup to modelize the action of control. The next step is to describe the evolution of controlled quantum system.

Usually in the literature, in order to obtain and justify the classical stochastic Schrödinger equations (1) and (2), Quantum Filtering theory [9] or Instrumental Process theory [7] are used. Such techniques are based on the Hilbertian formalism of Quantum Mechanics and on the theory of Stochastic Quantum Calculus. It uses heavy analytic machinery and all the subtleties of the non commutative character of quantum probability (conditionnal expectation in Von Neumann Algebra, partially observed system...). The starting point is the description of interaction between system and environment in terms of quantum stochastic differential equations (also called *Hudson Parthasarathy Equations* [23]). In order to apply such theory in the control setup, a theory in adequacy with the non commutative character have to be introduced. Even if it is satisfied, the derivation and the obtaining of stochastic Schrödinger equations with control is far from being obvious and intuitive (see [11]) and there are less rigorous results.

Recently, in the framework of the description of the interaction of a small system with environment (without measurement), in [4], the authors have introduced a discrete model of interactions: *"Quantum Repeated Interactions"*. The basic model is the one of a small system $\mathcal{H}_0$ in contact with an infinite chain of quantum system $\bigotimes_{j=1}^{\infty}\mathcal{H}$. Ones after the others each copy of $\mathcal{H}$ interacts with $\mathcal{H}_0$ during a time h.

Such approach of open quantum system yields a "good" and "useful" approximation model of continuous-time interaction models. Indeed by rescaling this interaction with respect to h, it is shown that models of interaction (described by stochastic quantum differential equations) can be obtained as continuous limits (h goes to zero) of discrete models. In the measurement setup, this approach has been adapted in [25] and [26]. In these articles, it is then shown that classical quantum trajectories (solutions of equations (1) and (2)) can be obtained as continuous limits of discrete models of quantum measurement called *"Quantum Repeated Measurements"* (also called *"Discrete Quantum Trajectory theory"*). The idea of discrete indirect measurements consists in performing a measurement of an observable of $\mathcal{H}$ after each interaction between $\mathcal{H}_0$ and a copy of $\mathcal{H}$. A discrete quantum trajectory is then a discrete random process describing the evolution of the state of $\mathcal{H}_0$ undergoing such repeated measurements. In this case, the approach of the theory of stochastic Schrödinger equations via approximation results is essentially based on classical probability theory (there are no problem of commutativity).

The main aim of this article is to adapt such technique in the framework of control. In this article, we present the notion of control in the model of quantum repeated interactions. In this setup, quantum repeated measurements give rise to a discrete models of indirect quantum measurement with control. By adapting convergence results of [25] and [26], we obtain the description of stochastic Schrödinger equations with control. With this approach, all the problem concerning non commutativity are avoided and the physical justification of stochastic models is rigorous and intuitive.

This article is structured as follows.

The first section is devoted to present discrete models of quantum measurement with control theory. We remember the mathematical model of quantum repeated interactions. Next, we introduce an appropriate notion of control in this setup and by introducing the measurement principle, we obtain the description of *discrete quantum trajectory with control*. Next, in order to prepare final convergence results, we adapt and enlarge the asymptotic assumptions presented in [4] to the context of control. To investigate such problems, we focus on a central case in physical application: a two-level atom in contact with a spin chain.

The second section is then devoted to continuous models. By applying the asymptotic assumptions on the two-level atom model of Section 1, we obtain two different discrete evolution equations (in asymptotic form) describing the evolution of the state of $\mathcal{H}_0$. Each evolution equation describes the evolution of a discrete quantum trajectory with control for a specific observable. For each equation, we investigate the continuous limit equation and we show the convergence.

In the last section, we present some applications of continuous models. In a first time, we investigate a model of a deterministic control: an atom monitored by a laser. By modeling a suitable interaction discrete model and by adapting the result of Section 2, we obtain a continuous stochastic model for this concrete example. In a second time we present a use of stochastic control: "Optimal Control theory". We adapt classical results concerning this theory in the quantum language.

1 Discrete Controlled Quantum Trajectories

This section is devoted to the presentation of the model of discrete quantum trajectories in presence of external control. Here, we present a natural way for modeling control theory.

1.1 Repeated Quantum Measurements with Control

In a first time, let us remember the general context of quantum repeated interactions.

A small system, represented by a Hilbert space $\mathcal{H}_0$, is in contact with an infinite chain of identical independent quantum systems. Each piece of the environment is represented by $\mathcal{H}$ and interacts, one after the others, with $\mathcal{H}_0$ during a time interval of length h. For example, a copy of $\mathcal{H}$ can represent an incoming photon or a measurement apparatus...

The space describing the first interaction between $\mathcal{H}_0$ and $\mathcal{H}$ is defined by the tensor product $\mathcal{H}_0 \otimes \mathcal{H}$. The evolution is given by a self-adjoint operator H_{tot} on the tensor product. This operator is called the total Hamiltonian and its general form is

$$H_{tot} = H_0 \otimes I + I \otimes H + H_{int}$$

where the operators H_0 and H are the free Hamiltonians of each system. The operator H_{int} represents the Hamiltonian of interaction. This allows to define a unitary-operator

$$U = e^{ih\,H_{tot}},$$

and the evolution of states of $\mathcal{H}_0 \otimes \mathcal{H}$, in the Schrödinger picture, is given by

$$\rho \mapsto U\,\rho\,U^\star.$$

After this first interaction, a second copy of $\mathcal{H}$ interacts with $\mathcal{H}_0$ in the same fashion and so on.

As the chain is supposed to be infinite, the whole sequence of interactions is described by the state space:

$$\mathbf{\Gamma} = \mathcal{H}_0 \otimes \bigotimes_{k \geq 1} \mathcal{H}_k \tag{3}$$

where $\mathcal{H}_k$ denotes the k-th copy of $\mathcal{H}$. The countable tensor product $\bigotimes_{k\geq 1} \mathcal{H}_k$ means the following. Consider that $\mathcal{H}$ is of finite dimension and that $\{X_0, X_1, \dots, X_n\}$ is a fixed orthonormal basis of $\mathcal{H}$. The orthogonal projector on $\mathbb{C}X_0$ is denoted by $|X_0\rangle\langle X_0|$. This is the ground state (or vacuum state) of $\mathcal{H}$. The tensor product is taken with respect to X_0 (for more details, see [4]).

The unitary evolution describing the k-th interaction is given by U_k which acts as U on $\mathcal{H}_0 \otimes \mathcal{H}_k$, whereas it acts like the identity operator on the other copies. If ρ is a state on $\mathbf{\Gamma}$, the effect of the k-th interaction is then:

$$\rho \mapsto U_k \, \rho \, U_k^\star$$

Hence the result of the k first interactions is described by the operator V_k on $\mathcal{B}(\mathbf{\Gamma})$ defined by the recursive formula:

$$\left\{ \begin{array}{ccc} V_{k+1} & = & U_{k+1} V_k \\ V_0 & = & I \end{array} \right. \tag{4}$$

and the evolution of states is given by

$$\rho \mapsto V_k \, \rho \, V_k^\star.$$

In this context, a main feature of this article is to present measurement and control theory. Let start by describing the control theory. An action of control consist in modifying the interaction at each new step depending on the previous step (this condition allows further to introduce stochastic control). Therefore if U_k is the unitary-operator describing the k-th interaction, it depends then on the time of interaction and on a parameter u_{k-1} which gives account of the control. Likewise this parameter depends on the interaction time; the operator U_k is then denoted by $U_k(h, u_{k-1}(h))$.

The whole sequence $\mathbf{u} = (u_k(h))$ is called the "control strategy". In term of $\mathbf{u}$, the k first interactions are then described by the unitary-operator $V_k^{\mathbf{u}}$:

$$V_k^{\mathbf{u}} = U_k(h, u_{k-1}(h)) \, U_{k-1}(h, u_{k-2}(h)) \, \ldots \, U_1(h, u_0(h)). \tag{5}$$

Finally, the evolution in presence of control is given by

$$\rho \mapsto V_k^{\mathbf{u}} \, \rho \, V_k^{\mathbf{u}\star}. \tag{6}$$

Before to give a complete definition of control strategies, we have to introduce the repeated measurement model.

Let us describe the basic procedure on each system of the chain. Let A be any observable on $\mathcal{H}_k$ with spectral decomposition $A = \sum_{j=0}^{p} \lambda_j P_j$, consider its natural ampliation as an observable on $\mathbf{\Gamma}$ by:

$$A^k := \bigotimes_{j=0}^{k-1} I \otimes A \otimes \bigotimes_{j \geq k+1} I \tag{7}$$

The accessible data are the eigenvalues of A^k and the result of the observation is random. If ρ is any state on $\mathbf{\Gamma}$, we observe λ_j with probability

$$P[\text{to observe } \lambda_j] = Tr[\, \rho \, P_j^k \,], \quad j \in \{0, \ldots, p\},$$

where the operator P_j^k corresponds to the ampliation of the eigenprojector P_j in the same way as (7). If we have observed the eigenvalue λ_j, the "projection" postulate called "wave packet reduction" imposes the state after the measurement to be

$$\rho_j = \frac{P_j^k\,\rho\,P_j^k}{Tr[\,\rho\,P_j^k\,]}.$$

Remark: This corresponds to the new reference state of our system. Another measurement of the same observable A^k (with respect to this state) should give $P[\text{to observe } \lambda_j] = 1$. Hence only one measurement give a significant information; it justifies a principle of repeated interactions.

Quantum repeated measurements with control are the combination of this previous principle and the successive interactions (6). After each interaction, a quantum measurement induces a random modification of the state of the system. It defines then a discrete process which is called "discrete controlled quantum trajectory". The description is as follows.

The initial state on $\mathbf{\Gamma}$ is chosen to be

$$\mu = \rho \otimes \bigotimes_{j\geq 1} \beta_j$$

where ρ is any state on $\mathcal{H}_0$ and each $\beta_i = \beta$ where β is any state on $\mathcal{H}$. The state after k interactions is denoted by $\mu_k^{\mathbf{u}}$, we have:

$$\mu_k^{\mathbf{u}} = V_k^{\mathbf{u}}\;\mu\;V_k^{\mathbf{u}\star}.$$

The probability space describing the experience is $\Sigma^{\mathbb{N}^\star}$ where $\Sigma = \{0,\ldots,p\}$. The integers i correspond to the indexes of the eigenvalues of A. We endow $\Sigma^{\mathbb{N}^\star}$ with the cylinder σ-algebra generated by the cylinder sets:

$$\Lambda_{i_1,\ldots,i_k} = \{\omega \in \Omega^{\mathbb{N}} / \omega_1 = i_1, \ldots, \omega_k = i_k\}.$$

Remarking that for all j, the unitary operator U_j commutes with all P^k for all $k < j$. For any set $\{i_1,\ldots,i_k\}$, we can define the following operator:

$$\begin{aligned}
\tilde{\mu}_k^{\mathbf{u}}(i_1,\ldots,i_k) &= I\otimes P_{i_1}\otimes\ldots\otimes P_{i_k}\otimes I\ldots\;\mu_k^{\mathbf{u}}\;I\otimes P_{i_1}\otimes\ldots\otimes P_{i_k}\otimes I\ldots \\
&= P_{i_k}^k\ldots P_{i_1}^1\;\mu_k^{\mathbf{u}}\;P_{i_1}^1\ldots P_{i_k}^k.
\end{aligned}$$

This is the non-normalized state corresponding to the successive observation of $\lambda_{i1},\ldots,\lambda_{ik}$. The probability to observe these eigenvalues is

$$P[\text{to observe } \lambda_{i1},\ldots,\lambda_{ik}] = Tr[\tilde{\mu}_k^{\mathbf{u}}(i_1,\ldots,i_k)].$$

By putting

$$P[\Lambda_{i_1,\ldots,i_k}] = P[\text{to observe } \lambda_{i1},\ldots,\lambda_{ik}],$$

it defines a probability measure on the cylinder sets of $\Sigma^{\mathbf{N}^\star}$ which satisfies the Kolmogorov Consistency Criterion. It defines then a unique probability measure on $\Sigma^{\mathbf{N}^\star}$. The discrete quantum trajectory with control strategy $\mathbf{u}$ on $\mathbf{\Gamma}$ is described by the following random sequence of states:

$$\begin{array}{rccc} \tilde{\rho}_k^{\mathbf{u}}: & \Sigma^{\mathbf{N}^\star} & \longrightarrow & \mathcal{B}(\mathbf{\Gamma}) \\ & \omega & \longmapsto & \tilde{\rho}_k^{\mathbf{u}}(\omega_1,\ldots,\omega_k) = \frac{\tilde{\mu}_k^{\mathbf{u}}(\omega_1,\ldots,\omega_k)}{Tr[\tilde{\mu}_k^{\mathbf{u}}(\omega_1,\ldots,\omega_k))]}. \end{array}$$

From this description, the following result is obvious.

Proposition 1 *Let* $\mathbf{u}$ *be any strategy and* $(\tilde{\rho}_k^{\mathbf{u}})$ *be the above random sequence of states we have for all* $\omega \in \Sigma^{\mathbb{N}}$*:*

$$\tilde{\rho}_{k+1}^{\mathbf{u}}(\omega) = \frac{P_{\omega_{k+1}}^{k+1}\, U_{k+1}(h,u_k(h)) \;\; \tilde{\rho}_k^{\mathbf{u}}(\omega) \;\; U_{k+1}^\star(h,u_k(h))\, P_{\omega_{k+1}}^{k+1}}{Tr\left[\, \tilde{\rho}_k^{\mathbf{u}}(\omega)\; U_{k+1}^\star(h,u_k(h))\, P_{\omega_{k+1}}^{k+1}\, U_{k+1}(h,u_k(h))\right]}.$$

Now at this stage, we can make precise the definition of control strategies which correspond to the case of deterministic or stochastic control mentioned in the Introduction.

Definition 1 *Let* $\mathbf{u} = (u_k(h))$ *be a control strategy and let* $(\tilde{\rho}_k^{\mathbf{u}})$ *be a quantum trajectory.*

1. *If there exists some function* u *from* $\mathbb{R}$ *to* $\mathbb{R}^n$ *such that for all* k :

$$u_k(h) = u(kh),$$

the control strategy is called deterministic. It is also called "open loop control".

2. *If there exists some function* u *from* $\mathbb{R} \times \mathcal{B}(\mathbf{\Gamma})$ *to* $\mathbb{R}^n$ *such that for all* k :

$$u_k(h) = u(kh, \tilde{\rho}_k^{\mathbf{u}}),$$

the control strategy is called Markovian. It is also called "closed loop control" or "feedback control". If for all k *we have* $u_k(h) = u(\tilde{\rho}_k^{\mathbf{u}})$*, this is an homogeneous Markovian strategy.*

The following theorem is an easy consequence of Proposition 1 and of the previous Definition.

Theorem 1 *For all control strategy* $\mathbf{u}$*, the sequence* $(\tilde{\rho}_n^{\mathbf{u}})_n$ *is a non homogeneous Markov chain valued on the set of states of* $\mathbf{\Gamma}$*. It is described as follows:*

$$P\left[\tilde{\rho}_{n+1}^{\mathbf{u}} = \mu / \tilde{\rho}_n^{\mathbf{u}} = \theta_n, \ldots, \tilde{\rho}_0^{\mathbf{u}} = \theta_0\right] = P\left[\tilde{\rho}_{n+1}^{\mathbf{u}} = \mu / \tilde{\rho}_n^{\mathbf{u}} = \theta_n\right].$$

If $\tilde{\rho}_n^{\mathbf{u}} = \theta_n$ *then* $\tilde{\rho}_{n+1}^{\mathbf{u}}$ *takes one of the values:*

$$\mathcal{H}_i^{\mathbf{u},n+1}(\theta_n) = \frac{P_i^{n+1}\, U_{n+1}(h,u_n(h))\;\theta_n\; U_{n+1}^\star(h,u_n(h))\, P_i^{n+1}}{Tr\left[U_{n+1}(h,u_n(h))\;\theta_n\; U_{n+1}^\star(h,u_n(h))\, P_i^{n+1}\right]}, \quad i = 0,\ldots,p,$$

with probability $Tr\left[U_{n+1}(h,u_n(h))\,\theta_n\, U_{n+1}^\star(h,u_n(h))\, P_i^{n+1}\right]$.

The discrete process $(\rho_k^{\mathbf{u}})$ *is called a controlled Markov chain.*

Proof: Property of being a Markov chain comes from the fact that a control strategy is either deterministic or Markovian. For the two cases, the conclusion is obvious from the description of Proposition 1. $\square$

In general, one is only interested in the reduced state of the small system. This state is given by the partial trace operation. Let us recall what partial trace is. Let $\mathcal{Z}$ be any Hilbert space, the notation $Tr_{\mathcal{Z}}[W]$ corresponds to the trace of any trace-class operator W on $\mathcal{Z}$.

Definition-Theorem 1 *Let $\mathcal{H}$ and $\mathcal{K}$ be any Hilbert spaces. Let α be a state on the tensor product $\mathcal{H}\otimes\mathcal{K}$. There exists a unique state η on $\mathcal{H}$ which is characterized by the property:*

$$Tr_{\mathcal{H}}[\,\eta\, X\,] = Tr_{\mathcal{H}\otimes\mathcal{K}}[\,\alpha(X\otimes I)\,].$$

for all $X\in\mathcal{B}(\mathcal{H})$. The state η is called the partial trace of α on $\mathcal{H}$ with respect to $\mathcal{K}$.

For any state α on $\mathbf{\Gamma}$, denote $\mathbf{E}_0[\alpha]$ the partial trace of α on $\mathcal{H}_0$ with respect to $\bigotimes_{k\geq 1}\mathcal{H}_k$. We then define the discrete controlled quantum trajectory on $\mathcal{H}_0$ as follows. For all $\omega\in\Sigma^{\mathbb{N}^\star}$:

$$\rho_n^{\mathbf{u}}(\omega) = \mathbf{E}_0[\tilde{\rho}_n^{\mathbf{u}}(\omega)]. \tag{8}$$

Remark: We adapt Definition 1 by considering Markovian strategy defined on $\mathbb{R}\times\mathcal{B}(\mathcal{H}_0)$. An immediate consequence of Theorem 1 is the following result.

Theorem 2 *For all control strategy $\mathbf{u}$, the random sequence defined by formula (8) is a non-homogeneous controlled Markov chain with values in the set of states on $\mathcal{H}_0$. If $\rho_n^{\mathbf{u}}=\chi_n$ then $\rho_{n+1}^{\mathbf{u}}$ takes one of the values:*

$$\mathbf{E}_0\left[\frac{I\otimes P_i\;\tilde{U}_{n+1}(h,u_n(h))\,(\chi_n\otimes\beta)\,\tilde{U}^\star_{n+1}(h,u_n(h))\;\;I\otimes P_i}{Tr[\,\tilde{U}_{n+1}(h,u_n(h))\,(\chi_n\otimes\beta)\,\tilde{U}^\star_{n+1}(h,u_n(h))\,I\otimes P_i]}\right]\quad i=0\dots p$$

with probability $Tr\left[\tilde{U}_{n+1}(h,u_n(h))(\chi_n\otimes\beta)\tilde{U}^\star_{n+1}(h,u_n(h))\;P_i\right]$.

Remark: Let us stress that:

$$\frac{(I\otimes P_i)\,U\,(\chi_n\otimes\beta)\,U^\star\,(I\otimes P_i)}{Tr[\,U\,(\chi_n\otimes\beta)\,U^\star\,(I\otimes P_i)]}$$

is a state on $\mathcal{H}_0\otimes\mathcal{H}$. In this situation, the notation $\mathbf{E}_0$ denotes the partial trace on $\mathcal{H}_0$ with respect to $\mathcal{H}$. Moreover for each n, the operator $\tilde{U}_n$, which appears in the description of the transition of the Markov chain $(\rho_n^{\mathbf{u}})$, acts on $\mathcal{H}_0\otimes\mathcal{H}$ as the operators U_n on $\mathcal{H}_0\otimes\mathcal{H}_n$

With the description of Theorem 2, we can express a discrete evolution equation describing the discrete quantum trajectory $(\rho_k^{\mathbf{u}})$. By putting

$$\mathcal{L}_i^{\mathbf{u},k}(\rho) = \mathbf{E}_0\left[I\otimes P_i\,\tilde{U}_k(h,u_{k-1}(h))\,(\rho\otimes\beta)\,\tilde{U}_k^\star(h,u_{k-1}(h))\,I\otimes P_i\right]\;\; i=0\dots p,$$

and $\mathbf{1}_i^k(\omega) = \mathbf{1}_i(\omega_k)$ for all $\omega \in \Sigma^{\mathbb{N}^\star}$, the discrete process $(\rho_k^{\mathbf{u}})$ then satisfies

$$\rho_{k+1}^{\mathbf{u}}(\omega) = \sum_{i=0}^{p} \frac{\mathcal{L}_i^{k+1}(\rho_k^{\mathbf{u}}(\omega))}{Tr[\mathcal{L}_i^{k+1}(\rho_k^{\mathbf{u}}(\omega))]} \mathbf{1}_i^{k+1}(\omega) \tag{9}$$

for all $\omega \in \Sigma^{\mathbb{N}}$ and all $k > 0$.

The following section is devoted to the deeply study of the equation (9) in a particular case of a two-level system in interaction with a spin chain. Next we come into the question of asymptotic assumptions.

1.2 A Two-Level Atom

The physical situation is described by $\mathcal{H}_0 = \mathcal{H} = \mathbb{C}^2$. In this case, an observable A has two different eigenvalues: $A = \lambda_0 P_0 + \lambda_1 P_1$ (the case which only one eigenvalue is not interesting). The equation (9) is reduced to:

$$\rho_{k+1}^{\mathbf{u}}(\omega) = \frac{\mathcal{L}_0^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}}(\omega))}{p_{k+1}^{\mathbf{u}}} \mathbf{1}_0^{k+1}(\omega) + \frac{\mathcal{L}_1^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}}(\omega))}{q_{k+1}^{\mathbf{u}}} \mathbf{1}_1^{k+1}(\omega). \tag{10}$$

where $p_{k+1}^{\mathbf{u}} = Tr[\mathcal{L}_0^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}})] = 1 - q_{k+1}^{\mathbf{u}}$. Let now introduce the centered and normalized random variable

$$X_{k+1} = \frac{\mathbf{1}_1^{k+1}(\omega) - q_{k+1}^{\mathbf{u}}}{\sqrt{q_{k+1}^{\mathbf{u}} p_{k+1}^{\mathbf{u}}}}.$$

We define the associated filtration on $\{0,1\}^{\mathbf{N}}$:

$$\mathcal{F}_k = \sigma(X_i, i \leq k).$$

So by construction we have $\mathbf{E}[X_{k+1}/\mathcal{F}_k] = 0$ and $\mathbf{E}[X_{k+1}^2/\mathcal{F}_k] = 1$. In terms of (X_k) the discrete controlled quantum trajectory satisfies:

$$\begin{aligned} \rho_{k+1}^{\mathbf{u}} \quad &= \quad \mathcal{L}_0^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}}) + \mathcal{L}_1^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}}) \\ &\quad + \left[-\sqrt{\frac{q_{k+1}^{\mathbf{u}}}{p_{k+1}^{\mathbf{u}}}} \mathcal{L}_0^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}}) + \sqrt{\frac{p_{k+1}^{\mathbf{u}}}{q_{k+1}^{\mathbf{u}}}} \mathcal{L}_1^{k+1}(\rho_k^{\mathbf{u}}) \right] X_{k+1}. \end{aligned} \tag{11}$$

To give more sense to the equation (11), we have to express the terms $\mathcal{L}_i^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}})$ in a more explicit way. For this, we introduce a specific basis. Let $(X_0 = \Omega, X_1 = X)$ be an orthonormal basis of $\mathcal{H}_0 = \mathcal{H} = \mathbb{C}^2$. For the space $\mathcal{H}_0 \otimes \mathcal{H}$, we consider the following basis

$$\Omega \otimes \Omega, X \otimes \Omega, \Omega \otimes X, X \otimes X.$$

In this basis, the unitary operator can be written by blocks as a 2×2 matrix:

$$U_{k+1}(h, u_k(h)) = \begin{pmatrix} L_{00}(kh, u_k(h)) & L_{01}(kh, u_k(h)) \\ L_{10}(kh, u_k(h)) & L_{11}(kh, u_k(h)) \end{pmatrix}$$

where each $L_{ij}(kh, u_k(h))$ are operators on $\mathcal{H}_0$. The reference state β of $\mathcal{H}$ is:

$$\beta = |\Omega\rangle\langle\Omega|.$$

The terms $\mathcal{L}_i^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}})$ depends also on the expression of the eigenprojectors of the observable A. If the eigenprojector P_i is expressed as $P_i = \begin{pmatrix} p_{00}^i & p_{01}^i \\ p_{10}^i & p_{11}^i \end{pmatrix}$ in the basis (Ω, X) of $\mathcal{H}$, we have:

$$\begin{aligned} \mathcal{L}_i^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}}) \; = \; & p_{00}^i L_{00}(kh, u_k(h))\, \rho_k^{\mathbf{u}}\, L_{00}^\star(kh, u_k(h)) + p_{01}^i L_{00}(kh, u_k(h))\, \rho_k^{\mathbf{u}}\, L_{10}^\star(kh, u_k(h)) \\ & + p_{10}^i L_{10}(kh, u_k(h))\, \rho_k^{\mathbf{u}}\, L_{00}^\star(kh, u_k(h)) + p_{11}^i L_{10}(kh, u_k(h))\, \rho_k^{\mathbf{u}}\, L_{10}^\star(kh, u_k(h)) \end{aligned} \tag{12}$$

As the unitary evolution depends on the time length interaction h, the discrete quantum trajectory $(\rho_k^{\mathbf{u}})$ depends on h. In Section 2, this dependence allows us to consider continuous time limit $(h \to 0)$ of the discrete processes $(\rho_k^{\mathbf{u}})$. The next section is devoted to present the asymptotic ingredients necessary to obtain such convergence results.

1.3 Description of Asymptotic

In this section, we present suitable asymptotic for the coefficients of the unitary operators $U_k(h, u_k(h))$ in order to have an effective continuous time limit from discrete quantum trajectories. Let $h = 1/n$ be the length time of interaction, we have for (U_k)

$$U_{k+1}(n, u_k(n)) = \begin{pmatrix} L_{00}(k/n, u_k(n)) & L_{01}(k/n, u_k(n)) \\ L_{10}(k/n, u_k(n)) & L_{11}(k/n, u_k(n)) \end{pmatrix},$$

In our context, the choice of the coefficients L_{ij} is an adaptation of the works of Attal-Pautrat in [4]. In their work, they consider only evolution of the type

$$U_{k+1}(n) = \begin{pmatrix} L_{00}(n) & L_{01}(n) \\ L_{10}(n) & L_{11}(n) \end{pmatrix},$$

that is, homogeneous evolution without control. They have shown that

$$V_{[nt]} = U_{[nt]}(n) \dots U_1(n)$$

converges (in operator algebra) to a non-trivial process V_t (solution of a quantum stochastic differential equation), only if the coefficients $L_{ij}(n)$ obey certain normalization. In their case, these coefficients must be of the form

$$L_{00}(n) = I + \frac{1}{n}\left(-iH_0 - \frac{1}{2}CC^\star\right) + \circ\left(\frac{1}{n}\right) \tag{13}$$

$$L_{10}(n) = \frac{1}{\sqrt{n}}C + \circ\left(\frac{1}{n}\right), \tag{14}$$

where H_0 is the Hamiltonian of $\mathcal{H}_0$ and C is any operator on $\mathbb{C}^2$. Hence, in the control context, the coefficients $L_{ij}(k/n, u_k(n))$ must follow similar expressions. Let k be fixed, we put

$$\begin{aligned} L_{00}(k/n, u_k(n)) &= I + \frac{1}{n}\left(-iH_k(n, u_k(n)) - \frac{1}{2}C_k(n, u_k(n))C_k^\star(n, u_k(n))\right) + \circ\left(\frac{1}{n}\right) \\ L_{00}(k/n, u_k(n)) &= \frac{1}{\sqrt{n}}C_k(n, u_k(n)) + \circ\left(\frac{1}{n}\right) \end{aligned} \quad (15)$$

where $H_k(n, u_k(n))$ is a self-adjoint operator and $C_k(n, u_k(n))$ is an operator on $\mathbb{C}^2$. It is straightforward that the expression (13) of Attal-Pautrat is a particular case of the previous expression. Finally, we suppose that there exist some function H and C such that

$$\begin{array}{llll} H: & \mathbb{R}^+ \times \mathbb{R} & \longrightarrow & \mathbb{H}_2(\mathbb{C}) \\ & (t, s) & \longmapsto & H(t, s) \end{array} \quad \text{and} \quad \begin{array}{llll} C: & \mathbb{R}^+ \times \mathbb{R} & \longrightarrow & \mathbb{M}_2(\mathbb{C}) \\ & (t, s) & \longmapsto & C(t, s) \end{array}$$

where $\mathbb{H}_2(\mathbb{C})$ designs the set of self-adjoint operators on $\mathbb{C}^2$ and

$$\begin{aligned} H_k(n, u_k(n)) &= H(k/n, u_k(n)) \\ C_k(n, u_k(n) &= C(k/n, u_k(n)) \end{aligned} \quad (16)$$

Furthermore we suppose that all the $\circ$ are uniform in k.

Now, we shall express the equation (11) and (12) with these asymptotic assumptions. As it was announced, we obtain two different behaviours depending of the choice of the observable.

1. If the observable A is diagonal in the basis (Ω, X), that is, it is of the form $A = \lambda_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we obtain the asymptotic for the probabilities

$$\begin{aligned} p_{k+1}^{\mathbf{u}}(n) &= 1 - \frac{1}{n}Tr\Big[\mathcal{J}(k/n, u_k(n))(\rho_k^{\mathbf{u}}(n))\Big] + \circ\left(\frac{1}{n}\right) \\ q_{k+1}^{\mathbf{u}}(n) &= \frac{1}{n}Tr\Big[\mathcal{J}(k/n, u_k(n))(\rho_k^{\mathbf{u}}(n))\Big] + \circ\left(\frac{1}{n}\right) \end{aligned}$$

The discrete equation (11) becomes

$$\begin{aligned} &\rho_{k+1}^{\mathbf{u}}(n) - \rho_k^{\mathbf{u}}(n) \\ &\quad = \frac{1}{n}L(k/n, u_k(n))(\rho_k^{\mathbf{u}}(n)) + \circ(\frac{1}{n}) \\ &\quad + \left[\frac{\mathcal{J}(k/n, u_k(n))(\rho_k^{\mathbf{u}}(n))}{Tr\Big[\mathcal{J}(k/n, u_k(n))(\rho_k^{\mathbf{u}}(n)))\Big]} - \rho_k^{\mathbf{u}}(n) + \circ(1)\right]\sqrt{q_{k+1}^{\mathbf{u}}(n)p_{k+1}^{\mathbf{u}}(n)}\, X_{k+1}(n) \end{aligned} \quad (17)$$

where for all state ρ, we have defined

$$\begin{aligned}\mathcal{J}(t,s)(\rho) &= C(t,s)\ \rho\ C^\star(t,s) \text{ and}\\ L(t,s)(\rho) &= -i[H(t,s),\rho]-\frac{1}{2}\{C(t,s)C^\star(t,s),\rho\}+\mathcal{J}(t,s)(\rho).\end{aligned}\tag{18}$$

2. If the observable A is non diagonal in the basis (Ω, X), and if the eigenprojectors are express as $P_0=\begin{pmatrix} p_{00} & p_{01}\\ p_{10} & p_{11}\end{pmatrix}$ and $P_1=\begin{pmatrix} q_{00} & q_{01}\\ q_{10} & q_{11}\end{pmatrix}$ we have

$$\begin{aligned}p^{\mathbf{u}}_{k+1} &= p_{00}+\frac{1}{\sqrt{n}}Tr\Big[\rho^{\mathbf{u}}_k\ \big(p_{01}C(k/n,u_{k+1}(n))+p_{10}C^\star(k/n,u_k(n))\big)\Big]\\ &\quad +\frac{1}{n}Tr\Big[\rho^{\mathbf{u}}_k\, p_{00}\ \big(C(k/n,u_k(n))+C^\star(k/n,u_k(n))\big)\Big]+\circ\left(\frac{1}{n}\right)\\ q^{\mathbf{u}}_{k+1} &= q_{00}+\frac{1}{\sqrt{n}}Tr\Big[\rho^{\mathbf{u}}_k\ \big(q_{01}C(k/n,u_k(n))+q_{10}C^\star(k/n,u_k(n))\big)\Big]\\ &\quad +\frac{1}{n}Tr\Big[\rho^{\mathbf{u}}_k\, q_{00}\ \big(C(k/n,u_k(n))+C^\star(k/n,u_k(n))\big)\Big]+\circ\left(\frac{1}{n}\right).\end{aligned}$$

The discrete equation (11) becomes

$$\begin{aligned}&\rho^{\mathbf{u}}_{k+1}-\rho^{\mathbf{u}}_k=\\ &\quad\frac{1}{n}L(k/n,u_k(n))(\rho^{\mathbf{u}}_k)+\circ\left(\frac{1}{n}\right)+\Big[e^{i\theta}C(k/n,u_k(n))\rho^{\mathbf{u}}_k+e^{-i\theta}\rho^{\mathbf{u}}_k C^\star(k/n,u_k(n))\\ &\quad -Tr\big[\rho^{\mathbf{u}}_k\ \big(e^{i\theta}C(k/n,u_k(n))+e^{-i\theta}C^\star(k/n,u_k(n))\big)\big]\ \rho^{\mathbf{u}}_k+\circ(1)\Big]\frac{1}{\sqrt{n}}X_{k+1}(n)\end{aligned}\tag{19}$$

where θ is a real parameter. This parameter can be explicitly expressed with the coefficients of the eigenprojectors (P_i). By putting $C_\theta(k/n,u_k(n))=e^{i\theta}C(k/n,u_k(n))$ we have the same form for the equation (19) for all θ, then we consider in the following that $\theta=0$. The expression of L is the same as (18).

In order to prepare the final convergence result, in each case, we can define a process $(\rho_{[nt]})$ which satisfies

$$\begin{aligned}\rho^{\mathbf{u}}_{[nt]} &= \rho_0+\sum_{i=0}^{[nt]-1}[\rho^{\mathbf{u}}_{i+1}-\rho^{\mathbf{u}}_i]\\ &= \rho_0+\sum_{i=0}^{[nt]-1}[\mathcal{L}^{\mathbf{u},i+1}_0(\rho^{\mathbf{u}}_i)+\mathcal{L}^{\mathbf{u},i+1}_1(\rho^{\mathbf{u}}_i)-\rho^{\mathbf{u}}_i]\\ &\quad +\sum_{i=0}^{[nt]-1}\left[-\sqrt{\frac{q^{\mathbf{u}}_{i+1}}{p^{\mathbf{u}}_{i+1}}}\,\mathcal{L}^{\mathbf{u},i+1}_0(\rho^{\mathbf{u}}_i)+\sqrt{\frac{p^{\mathbf{u}}_{i+1}}{q^{\mathbf{u}}_{i+1}}}\,\mathcal{L}^{\mathbf{u},i+1}_1(\rho^{\mathbf{u}}_i)\right]X_{i+1}\\ &= \rho_0+\sum_{i=0}^{[nt]-1}\frac{1}{n}\mathcal{Y}(i/n,u_i(n),\rho^{\mathbf{u}}_i)+\sum_{i=0}^{[nt]-1}\mathcal{Z}(i/n,u_i(n),\rho^{\mathbf{u}}_i)X_{i+1}\end{aligned}\tag{20}$$

for some functions $\mathcal{Y}$ and $\mathcal{Z}$ which depend on the description (17) or (19).

Depending on the choice of observable, in the next section, we show that the process $(\rho_{[nt]})$ converges to a solution of a particular stochastic differential equation.

2 Convergence to Continuous Models

In this section, starting from the description (20) with a Markovian strategy and following the asymptotic (17) and (19), we show that discrete processes $(\rho_{[nt]})$ converge in distribution to solutions of stochastic differential equations.

As in the classical case of stochastic differential equations, we show that the evolution of a quantum system undergoing a continuous measurement with control is either described by a diffusive evolution or by an evolution with jump.

1. If (ρ_t) denotes the state of a quantum system, the diffusive evolution is given by
$$d\rho_t = L\big(t, u(t, \rho_t)\big)(\rho_t)dt + \Theta\big(t, u(t, \rho_t)\big)(\rho_t)dW_t \tag{21}$$
where (W_t) describes a one-dimensional Brownian motion. The function L is expressed as (18) and Θ is defined by
$$\Theta(t, a)(\mu) = C(t, a)\mu + \mu C^\star(t, a) - Tr\left[\mu\Big(C(t, a) + C^\star(t, a)\Big)\right]\mu \tag{22}$$
for all $t > 0$, for all a in $\mathbb{R}$ and all operator μ in $\mathbb{M}_2(\mathbb{C})$.

2. The evolution with jump is given by
$$\begin{aligned} d\rho_t &= L\big(t, u(t, \rho_t)\big)(\rho_t)dt \\ &\quad + \left[\frac{\mathcal{J}\big(t, u(t, \rho_t)\big)(\rho_t)}{Tr\Big[\mathcal{J}\big(t, u(t, \rho_t)\big)(\rho_t)\Big]} - \rho_t\right]\left(d\tilde{N}_t - Tr\Big[\mathcal{J}\big(t, u(t, \rho_t)\big)(\rho_t)\Big]dt\right) \end{aligned} \tag{23}$$
where $\tilde{N}_t$ is a counting process with stochastic intensity $\int_0^t Tr\Big[\mathcal{J}\big(s, u(s, \rho_s)\big)(\rho_s)\Big]ds$. The functions L and $\mathcal{J}$ are as (18).

In a natural way, we call such equations *"Controlled Stochastic Schrödinger Equations"* and their solutions *"Controlled Quantum Trajectories"*.

For the moment, we do not speak about the regularity of the functions L, Θ and $\mathcal{J}$. This will be discussed when we deal with the question of existence and uniqueness of a solution for such equations.

This question of existence and uniqueness is relatively important because this problem is not really treated in details in the literature. Moreover in the two cases, it is difficult to show that a solution of these equations is valued in the set of states (actually it is a essential property to solve the problem of existence and uniqueness). For physical considerations, this property is crucial otherwise these equations have no sense; we are going to see that this point can be deduced from the convergence result. Let us start by studying the diffusive case.

2.1 Diffusive Equation with Control

In this section, we justify the diffusive model

$$d\rho_t = L\big(t, u(t,\rho_t)\big)(\rho_t)dt + \Theta\big(t, u(t,\rho_t)\big)(\rho_t)dW_t$$

of controlled stochastic Schrödinger equations by proving that the solution of equation (21) is obtained from the limit of particular quantum trajectories $(\rho_{[nt]})$. In the same time, we show that the equation (21) admits a unique solution with values in the set of states.

Start by investigating the problem of existence and uniqueness of a solution for (21). For the moment, let u be any measurable function which defines a Markovian strategy as it is expressed in Definition 2. Usual conditions concerning existence and uniqueness of a solution for SDE of type (21) is that for all $T > 0$ there exists a constant $M(T)$ and $K(T)$ such that the function L and Θ satisfy for all $t \leq T$ and $(\mu, \rho) \in \mathbb{M}_2(\mathbb{C})^2$:

$$\begin{aligned} &\sup\big\{\|L(t,a)(\mu) - L(t,a)(\rho)\|, \|\Theta(t,a)(\mu) - \Theta(t,a)(\rho)\|\big\} \leq K(T)\|\mu - \rho\| \\ &\sup\big\{\|L(t,a)(\rho)\|, \|\Theta(t,a)(\rho)\|\big\} \leq M(T)(1 + \|\rho\| + \|a\|). \end{aligned} \tag{24}$$

Such conditions is called global Lipschitz conditions. However even in the homogeneous case without control, such conditions are not satisfied. Indeed, in the homogeneous situation without control, for Θ we have

$$\Theta(t,a)(\mu) = \Theta(\mu) = C\mu + \mu C^\star - Tr\left[\mu\big(C + C^\star\big)\right]\mu.$$

Such function is not Lipschitz. Nevertheless it is C^∞ and then local Lipschitz. Such property is used in the classical case to obtain the existence and the uniqueness of a solution for stochastic Schrödinger equations (see [25] and [26]). In the non-homogeneous context with control, the local Lipschitz condition is expressed as follows. For all integer $k > 0$ and all $x \in \mathbb{R}$, define the function ϕ^k by

$$\phi^k(x) = -k\mathbf{1}_{]-\infty,-k[}(x) + x\mathbf{1}_{[-k,k]}(x) + k\mathbf{1}_{]k,\infty[}(x).$$

The function ϕ^k is called a truncation function. Its extension on the set of operator on $\mathbb{C}^2$ is given by

$$\tilde{\phi}^k(B) = (\phi^k(Re(B_{ij})) + i\phi^k(Im(B_{ij})))_{0\leq i,j\leq 1}.$$

Hence, the local Lipschitz condition for the functions L and θ can be expressed as follows. For all $T > 0$ and for all integer $k > 0$ there exists a constant $M^k(T)$ and $K^k(T)$ such that the function L and Θ satisfy for all $t \leq T$ and $(\mu, \rho) \in \mathbb{M}_2(\mathbb{C})^2$:

$$\begin{aligned} &\|L(t,a)(\tilde{\phi}^k(\mu)) - L(t,a)(\tilde{\phi}^k(\rho))\| \leq K^k(T)\|\mu - \rho\| \\ &\|\Theta(t,a)(\tilde{\phi}^k(\mu)) - \Theta(t,a)(\tilde{\phi}^k(\rho))\| \leq K^k(T)\|\mu - \rho\| \\ &\sup\big\{\|L(t,a)(\tilde{\phi}^k(\rho))\|, \|\Theta(t,a)(\tilde{\phi}^k(\rho))\|\big\} \leq M^k(T)(1 + \|\rho\| + \|a\|). \end{aligned} \tag{25}$$

As a consequence, we have the following existence and uniqueness theorem.

Theorem 3 *Let u be any measurable function. Let $k > 0$ be an integer. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space which supports a standard Brownian motion (W_t). Assume that L and Θ satisfy the conditions (25). Let ρ_0 be any 2×2 matrix. The stochastic differential equation*

$$\rho_t^{\mathbf{u},k} = \rho_0 + \int_0^t L\big(s, u(s, \tilde{\phi}^k(\rho_s^{\mathbf{u},k})\big)(\tilde{\phi}^k(\rho_s^{\mathbf{u},k}))ds + \int_0^t \Theta\big(s, u(s, \tilde{\phi}^k(\rho_s^{\mathbf{u},k}))\big)(\tilde{\phi}^k(\rho_s^{\mathbf{u},k}))dW_s, \quad (26)$$

admits a unique solution $(\rho_t^{\mathbf{u},k})$. Furthermore the application $t \mapsto \rho_t^{\mathbf{u},k}$ is almost surely continuous.

This theorem is just a consequence of the local Lipschitz condition (25) (cf [29]). The process $(\rho_t^{\mathbf{u},k})$ is called a truncated solution. The link between such solution and a solution of the equation (21) without truncature is expressed as follows. Usually, we define the random stopping time

$$T_k = \inf\{t > 0 / \exists (ij),\ Re(\rho_t^{\mathbf{u},k}(ij)) = k \text{ or } Im(\rho_t^{\mathbf{u},k}(ij)) = k\}$$

For any $k > 1$, we have $T_k > 0$ almost surely for ρ_0 is a state and the almost surely continuity of $(\rho_t^{\mathbf{u},k})$ (the coefficients of ρ_0 satisfy namely $|\rho_0(ij)| \leq 1$). Furthermore on $[0, T_k[$ we have

$$\tilde{\phi}^k(\rho_t^{\mathbf{u},k}) = \rho_t^{\mathbf{u},k}.$$

Therefore the process $(\rho_t^{\mathbf{u},k})$ satisfies on $[0, T_k[$

$$\rho_t^{\mathbf{u},k} = \rho_0 + \int_0^t L\big(s, u(s, \rho_s^{\mathbf{u},k})\big)(\rho_s^{\mathbf{u},k})ds + \int_0^t \Theta\big(s, u(s, \rho_s^{\mathbf{u},k})\big)(\rho_s^{\mathbf{u},k})dW_s, \quad (27)$$

Hence the process $(\rho_t^{\mathbf{u},k})$ solution of (26) is the unique solution of the equation (21) on $[0, T_k[$.

In our situation, we will prove that $T_k = \infty$ for all $k > 1$ by proving that the process $(\rho_t^{\mathbf{u},k})$ is valued in the set of states. Indeed if the process $(\rho_t^{\mathbf{u},k})$ takes value in the set of states, we have for all $t \geq 0$

$$\tilde{\phi}^k(\rho_t^{\mathbf{u},k}) = \rho_t^{\mathbf{u},k},$$

then $T_k = \infty$. As a consequence the process $(\rho_t^{\mathbf{u},k})$ satisfies for all $t > 0$ the equation (21). The truncature method becomes actually not necessary, it just allow to exhibit a solution. As a consequence, we have to prove that the solution obtained with a truncature method takes value in the set of states. This property follows from the convergence theorem.

Indeed, let assume that there is a discrete quantum trajectory $(\rho_{[nt]}^{\mathbf{u}})$ which converges in distribution to $(\rho_t^{\mathbf{u},k})$ (for some $k > 1$). Such convergence is denoted by

$$\rho_{[nt]}^{\mathbf{u}} \Longrightarrow \rho_t^{\mathbf{u},k}.$$

Therefore for all measurable functions $\mathcal{V}$ defined on $\mathbb{M}_2(\mathbb{C})$, we have

$$\mathcal{V}(\rho_{[nt]}^{\mathbf{u}}) \Longrightarrow \mathcal{V}(\rho_t^{\mathbf{u},k})$$

We apply it for the functions $\mathcal{V}(\rho)=Tr[\rho]$, for $\mathcal{V}(\rho)=\rho^\star-\rho$ and $\mathcal{V}_z(\rho)=\langle z,\rho z\rangle$ for all $z\in\mathbb{C}^2$. By definition if ρ is a state we have from trace property $Tr[\rho]=1$, from self-adjointness $\rho^\star-\rho=0$ and from positivity $\langle z,\rho z\rangle\geq 0$ for all $z\in\mathbb{C}^2$. As discrete quantum trajectories take values in the set of states, these properties are then conserved at the limit. The limit process $\rho_t^{\mathbf{u},k}$ takes then also values in the set of states. In [26], the problem of existence and uniqueness is proved independently of the convergence result. In this case, using convergence result is more practical because the equation is more complicated. Let us prove now the convergence result.

Back to the description (20) of discrete quantum trajectories, with asymptotic (19) in the case of a non-diagonal observable A and with a Markovian strategy, we have

$$\begin{aligned}\rho^{\mathbf{u}}_{[nt]} &= \rho_0+\sum_{k=1}^{[nt]-1}\frac{1}{n}\Big[L\big(k/n,u(k/n,\rho^{\mathbf{u}}_k)\big)(\rho^{\mathbf{u}}_k)+\circ(1)\Big] \\ &\quad+\sum_{k=1}^{[nt]-1}\Big[\Theta\big(k/n,u(k/n,\rho^{\mathbf{u}}_k)\big)(\rho^{\mathbf{u}}_k)+\circ(1)\Big]\frac{1}{\sqrt{n}}X_{k+1}(n).\end{aligned}\tag{28}$$

From this description, we can define the following processes and functions:

$$\begin{aligned} W_n(t) &= \frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]}X_k(n)\\ V_n(t) &= \frac{[nt]}{n}\\ \rho^{\mathbf{u}}_n(t) &= \rho^{\mathbf{u}}_{[nt]}(n)\\ u_n(t,W) &= u([nt]/n,W)\\ \Theta_n(t,s) &= \Theta([nt]/n,s)\\ L_n(t,s) &= L([nt]/n,s)\end{aligned}\tag{29}$$

for all $t>0$, for all $s\in\mathbb{R}$ and for all $W\in\mathbb{M}_2(\mathbb{C})$.

By observing that these processes and these functions are piecewise constant, we can describe the discrete quantum trajectory $(\rho^{\mathbf{u}}_n(t))$ as a solution of the following stochastic differential equation

$$\begin{aligned}\rho^{\mathbf{u}}_n(t) &= \rho_0+\int_0^t\Big[L_n\Big(s_-,u_n(s_-,\rho^{\mathbf{u}}_n(s_-))\Big)(\rho^{\mathbf{u}}_n(s_-))+\circ(1)\Big]\,dV_n(s)\\ &\quad+\int_0^t\Big[\Theta_n\Big(s_-,u_n(s_-,\rho^{\mathbf{u}}_n(s_-))\Big)(\rho^{\mathbf{u}}_n(s_-))+\circ(1)\Big]\,dW_n(s)\\ &= \rho_0+\int_0^t\Big[L_n\Big(s_-,u_n(s_-,\tilde{\phi}^k(\rho^{\mathbf{u}}_n(s_-))\Big)(\tilde{\phi}^k(\rho^{\mathbf{u}}_n(s_-)))+\circ(1)\Big]\,dV_n(s)\\ &\quad+\int_0^t\Big[\Theta_n\Big(s_-,u_n(s_-,\tilde{\phi}^k(\rho^{\mathbf{u}}_n(s_-))\Big)(\tilde{\phi}^k(\rho^{\mathbf{u}}_n(s_-)))+\circ(1)\Big]\,dW_n(s)\end{aligned}\tag{30}$$

for all $k > 1$. It appears essentially as a discrete version of equation (21). This procedure was also used in [26], but the equation without control is more simple and the convergence result needs less assumptions and arguments.

Let us present the arguments in the control framework. In order to prove the convergence of this process to the solution of the equation (26) given by Theorem 3, we use a results of Kurtz and Protter ([20], [21]) concerning weak convergence of stochastic integrals. Let us fix some notations.

For all $T > 0$ we define $\mathcal{D}[0, T]$ the space of càdlàg process of $\mathbb{M}_2(\mathbb{C})$ endowed with the Skorohod topology.

Let $T_1[0, \infty)$ denote the set of non decreasing mapping λ from $[0, \infty)$ to $[0, \infty)$ with $\lambda(0) = 0$ such that $\lambda(t + h) - \lambda(t) \leq h$ for all $t, h \geq 0$. For any function G defined from $\mathbb{R}^+ \times \mathbb{M}_2(\mathbb{C})$ to $\mathbb{M}_2(\mathbb{C})$, we define

$$\begin{array}{rccc} \tilde{G}: & \mathcal{D}[0,\infty) \times T_1[0,\infty) & \longrightarrow & \mathcal{D}[0,\infty) \\ & (X, \lambda) & \longmapsto & G(X) \circ \lambda, \end{array}$$

such that for all $t \geq 0$ we have $G(X) \circ \lambda(t) = G(\lambda(t), X_{\lambda(t)})$. We consider the same definition for all other functions. We introduce the two following conditions concerning a function $\tilde{G}$ and a sequence $\tilde{G}_n$ as above.

$(C1)$ For each compact subset $\mathcal{K} \in \mathcal{D}[0, \infty) \times T_1[0, \infty)$ and $t > 0$,
$\sup_{(X,\lambda)} \sup_{s \leq t} \|\tilde{G}_n(X, \lambda)(s) - \tilde{G}(X, \lambda)(s)\| \to 0.$

$(C2)$ For $(X_n, \lambda_n)_n \in \mathcal{D}[0, \infty) \times T_1[0, \infty) / \sup_{s \leq T} \|X_n(s) - X(s)\| \to 0$
and $\sup_{s \leq t} |\lambda_n(s) - \lambda(s)| \to 0$ for each $t > 0$ implies

$$\sup_{s \leq t} \|\tilde{G}(X_n, \lambda_n)(s) - \tilde{G}(X, \lambda)(s)\| \to 0. \tag{31}$$

Furthermore, recall that the square-bracket $[X, X]$ is defined for a semi-martingale by the formula:

$$[X, X]_t = X_t^2 - 2\int_0^t X_{s-} dX_s \,.$$

We shall denote by $T_t(V)$ the total variation of a finite variation processes V on the interval $[0, t]$. The Theorem of Kurtz and Protter [21] that we use is the following.

Theorem 4 *Let (H_n, H) and (K_n, K) be two couple of functions which satisfy the conditions $(C1)$ and $(C2)$. Let $(\mathcal{F}_t^n)$ be a filtration and let $X_n(t)$ be a $\mathcal{F}_t^n$-adapted process which satisfies*

$$X_n(t) = X(0) + \int_0^t H_n(s, X_n(s_-)) dV_n(s) + \int_0^t K_n(s, X_n(s_-)) dW_n(s). \tag{32}$$

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space. Let X_t be the unique solution of

$$X_n(t) = X(0) + \int_0^t H(s, X_s) ds + \int_0^t K(s, X_s) dW_s \tag{33}$$

where (W_t) *is a standard Brownian motion on* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.

Suppose that (W_n, V_n) *converges in distribution in the Skorohod topology to* (W, V) *where* $V_t = t$ *for all* $t \geq 0$ *and suppose*

$$\sup_n \left\{ \mathbf{E}^n \Big[[W_n, W_n]_t \Big] \right\} < \infty, \tag{34}$$

$$\sup_n \left\{ \mathbf{E}^n \left[T_t(V_n) \right] \right\} < \infty. \tag{35}$$

Hence the process $(X_n(t))$ *converges in distribution in* $\mathcal{D}[0, T]$ *for all* $T > 0$ *to the process* (X_t).

We wish then to apply this theorem to obtain the convergence result for discrete quantum trajectories $(\rho_{[nt]})$ described by (30). Concerning the convergence of the processes (W_n) and V_n we use the following theorem which is a generalization of Donsker Theorem (see [31]).

Theorem 5 *Let* (M_n) *be a sequence of martingales. Suppose that*

$$\lim_{n \to \infty} \mathbf{E}\big[|[M_n, M_n]_t - t| \big] = 0.$$

Then M_n *converges in distribution to a standard Brownian motion.*

In our context we have the following proposition.

Proposition 2 *Let* $(\mathcal{F}_t^n)$ *be the filtration*

$$\mathcal{F}_t^n = \sigma(X_i, i \leq [nt]). \tag{36}$$

The process $(W_n(t))$ *defined by* (29) *is a* $\mathcal{F}_t^n$*-martingale. We have*

$$W_n(t) \Longrightarrow W_t$$

where (W_t) *is a standard Brownian motion.*

Moreover we have

$$\sup_n \mathbf{E}\Big[[W_n, W_n]_t \Big] < \infty\,.$$

Finally, we have the convergence in distribution for the process $(W_n, V_n,)$ *to* (W, V) *when* n *goes to infinity.*

Proof: Thanks to the definition of the random variable X_i, we have $\mathbf{E}[X_{i+1}/\mathcal{F}_i^n] = 0$ which implies $\mathbf{E}\left[\sum_{i=[ns]+1}^{[nt]} X_i/\mathcal{F}_s^n\right] = 0$ for $t > s$. Thus if $t > s$ we have the martingale property:

$$\mathbf{E}[W_n(t)/\mathcal{F}_s^n] = W_n(s) + \mathbf{E}\left[\frac{1}{\sqrt{n}} \sum_{i=[ns]+1}^{[nt]} X_i/\mathcal{F}_s^n \right] = W_n(s).$$

By definition of $[Y, Y]$ for a stochastic process we have

$$[W_n, W_n]_t = W_n(t)^2 - 2\int_0^t W_n(s_-)dW_n(s) = \frac{1}{n}\sum_{i=1}^{[nt]} X_i^2.$$

Thus we have

$$\begin{aligned}\mathbf{E}\big[[W_n, W_n]_t\big] &= \frac{1}{n}\sum_{i=1}^{[nt]}\mathbf{E}[X_i^2] = \frac{1}{n}\sum_{i=1}^{[nt]}\mathbf{E}[\mathbf{E}[X_i^2/\sigma\{X_l, l<i\}]] \\ &= \frac{1}{n}\sum_{i=1}^{[nt]} 1 = \frac{[nt]}{n}.\end{aligned}$$

Hence we have

$$\sup_n \mathbf{E}\big[[W_n, W_n]_t\big] \leq t < \infty\,.$$

Let us prove the convergence of (W_n) to (W). According to Theorem 5, we must prove that

$$\lim_{n\to\infty}\mathbf{E}\big[|[M_n, M_n]_t - t|\big] = 0.$$

Actually we prove a L_2 convergence:

$$\lim_{n\to\infty}\mathbf{E}\big[|[M_n, M_n]_t - t|^2\big] = 0,$$

which implies the L_1 convergence. In order to show this convergence, we use the following property

$$\mathbf{E}\left[X_i^2\right] = \mathbf{E}\left[\mathbf{E}[X_i^2/\sigma\{X_l, l<i\}]\right] = 1$$

and if $i<j$

$$\begin{aligned}\mathbf{E}\left[(X_i^2-1)(X_j^2-1)\right] &= \mathbf{E}\left[(X_i^2-1)(X_j^2-1)/\sigma\{X_l, l<j\}]\right] \\ &= \mathbf{E}\left[(X_i^2-1)\right]\mathbf{E}\left[(X_j^2-1)\right] \\ &= 0\,.\end{aligned}$$

This gives

$$\begin{aligned}\mathbf{E}\left[\left([W_n, W_n]_t - \frac{[nt]}{n}\right)^2\right] &= \frac{1}{n^2}\sum_{i=1}^{[nt]}\mathbf{E}\left[(X_i^2-1)^2\right] + \frac{1}{n^2}\sum_{i<j}\mathbf{E}\left[(X_i^2-1)(X_j^2-1)\right] \\ &= \frac{1}{n^2}\sum_{i=1}^{[nt]}\mathbf{E}\left[(X_i^2-1)^2\right].\end{aligned}$$

Thanks to the fact that p_{00} and q_{00} are not equal to zero (because the observable A is not diagonal!) the terms $\mathbf{E}\left[(X_i^2-1)^2\right]$ are bounded uniformly in i so we have:

$$\lim_{n\to\infty}\mathbf{E}\left[\left([W_n, W_n]_t - \frac{[nt]}{n}\right)^2\right] = 0\,.$$

As $\frac{[nt]}{n} \longrightarrow t$ in L_2 we have the desired convergence. The convergence of (W_n, V_n) is then straightforward. $\square$

In order to conclude to the convergence result by using Theorem 4 of Kurtz and Protter, we have to verify conditions $(C1)$ and $(C2)$ for the functions appearing in the equation (30). We consider $\tilde{L}_n$ defined by

$$\tilde{L_n}(X) \circ (\lambda)(t) = L_n(\lambda(t), u_n(\lambda(t), X_{\lambda(t)}))(X_{\lambda(t)}) + \circ(1)$$

for all $t > 0$, for all $\lambda \in T_1[0, \infty)$ and all càdlàg process (X_t). Let us stress that in restriction to the processes (ρ_t) which takes values in the set of states, the $\circ$ are uniform in (ρ_t), we can then consider that the $\circ$ are uniform for all processes. We define $\tilde{\Theta}_n$ in the same way.

With this notation, we can express the convergence theorem.

Theorem 6 *Let $\mathcal{F}_t^n$ be the filtration defined by* (36). *Let ρ_0 be any state on $\mathcal{H}_0$. Let $(\rho_n^{\mathbf{u}}(t))$ be the discrete quantum trajectory satisfying:*

$$\begin{aligned} \rho_n^{\mathbf{u}}(t) \quad &= \quad \rho_0 + \int_0^t \left[L_n(s_-, u_n(s_-, \rho_n^{\mathbf{u}}(s_-)))(, \rho_n^{\mathbf{u}}(s_-)) + \circ(1)\right] dV_n(s) \\ &\quad + \int_0^t \left[\Theta_n(s_-, u_n(s_-, \rho_n^{\mathbf{u}}(s_-)))(\rho_n^{\mathbf{u}}(s_-)) + \circ(1)\right] dW_n(s). \end{aligned} \tag{37}$$

Let $k > 1$ be any integer. Let $(\rho_t^{\mathbf{u},k})$ be the unique solution of

$$\rho_t^{\mathbf{u},k} = \rho_0 + \int_0^t L(s, u(s, \tilde{\phi}^k(\rho_s^{\mathbf{u},k}))(\tilde{\phi}^k(\rho_s^{\mathbf{u},k}))ds + \int_0^t \Theta(s, u(s, \tilde{\phi}^k(\rho_s^{\mathbf{u},k})))(\tilde{\phi}^k(\rho_s^{\mathbf{u},k}))dW_s. \tag{38}$$

Assume the function u is sufficiently regular such that $\tilde{L}_n$, $\tilde{\Theta}_n$, $\tilde{L}$ and $\tilde{\Theta}$ composed with the truncature function $\tilde{\phi}^k$ satisfy conditions $(C1)$ and $(C2)$.

Then for all $T > 0$, the process $(\rho_n^{\mathbf{u}}(t))$ converges in distribution in $\mathcal{D}[0, T]$ to the process (ρ_t).

Finally the process $(\rho_t^{\mathbf{u}})$ is the unique solution of the controlled diffusive equation

$$\rho_t^{\mathbf{u}} = \rho_0 + \int_0^t L(s, u(s, \rho_s^{\mathbf{u}})(\rho_s^{\mathbf{u}})ds + \int_0^t \Theta(s, u(s, \rho_s^{\mathbf{u},k}))(\rho_s^{\mathbf{u}})dW_s. \tag{39}$$

Proof: As the condition $(C1)$ and $(C2)$ are assumed to be satisfied, thanks to Proposition 2 and Theorem 4, we have the convergence result. The final part of the theorem comes from the fact that the property of being a state is conserved by passage to the limit (see the remark at the beginning of this section). $\square$

As regards conditions $(C1)$ and $(C2)$, the assumption for the function u is satisfied for example when u is continuous. By definition of the functions L_n and Θ_n conditions $(C1)$ and $(C2)$ are namely satisfied for the functions L and Θ satisfy the local Lipschitz conditions (25) (used in Theorem 3 of existence and uniqueness).

Hence, the model of diffusive stochastic differential equation (21) for continuous measurement with control is physically justified by proving that solutions of such equations are obtained by limit of concrete discrete procedures. In the next section, we show a similar result by considering continuous limit of discrete quantum trajectories of type (17).

2.2 Jump Equation with Control

In this section, we investigate the convergence of a discrete quantum trajectory which comes from repeated measurements of a diagonal observable.

In all this section, we fix a strategy $\mathbf{u}$ which defines a Markovian strategy. Furthermore, as in the diffusive case, we suppose that this strategy is continuous. Let A be any diagonal observable. With the use of description (17) and (20), the discrete quantum trajectory satisfies

$$\begin{aligned}\rho^{\mathbf{u}}_{[nt]} &= \rho_0+\sum_{k=0}^{[nt]-1}\frac{1}{n}\Big[L\big(k/n,u(k/n,\rho^{\mathbf{u}}_k)\big)(\rho^{\mathbf{u}}_k)-\mathcal{J}\big(k/n,u(k/n,\rho^{\mathbf{u}}_k)\big)(\rho^{\mathbf{u}}_k)\\ &\quad+Tr\big[\mathcal{J}\big(k/n,u(k/n,\rho^{\mathbf{u}}_k)\big)(\rho^{\mathbf{u}}_k)\big]\,\rho^{\mathbf{u}}_k+\circ(1)\Big]\\ &\quad+\sum_{k=0}^{[nt]-1}\left[\frac{\mathcal{J}\big(k/n,u(k/n,\rho^{\mathbf{u}}_k)\big)(\rho^{\mathbf{u}}_k)}{Tr\big[\mathcal{J}(k/n,u(k/n,\rho^{\mathbf{u}}_k))(\rho^{\mathbf{u}}_k)\big]}-\rho^{\mathbf{u}}_k+\circ(1)\right]\nu_{k+1}.\end{aligned} \tag{40}$$

Following the idea presented in article [25], we aim to show that the process $(\rho_{[nt]})$ converges $(n\to\infty)$ to a process (ρ_t) which satisfies

$$\begin{aligned}\rho^{\mathbf{u}}_t &= \rho_0+\int_0^t\Big[L\big(s_-,u(s_-,\rho^{\mathbf{u}}_{s-})\big)(\rho^{\mathbf{u}}_{s-})+Tr\big[\mathcal{J}\big(s_-,u(s_-,\rho^{\mathbf{u}}_{s-})\big)(\rho^{\mathbf{u}}_{s-})\big]\,\rho^{\mathbf{u}}_{s-}\\ &\qquad-\mathcal{J}\big(s_-,u(s_-,\rho^{\mathbf{u}}_{s-})\big)(\rho^{\mathbf{u}}_{s-})\Big]ds \qquad\qquad (41)\\ &\quad+\int_0^t\int_{\mathbb{R}}\left[\frac{\mathcal{J}\big(s_-,u(s_-,\rho^{\mathbf{u}}_{s-})\big)(\rho^{\mathbf{u}}_{s-})}{Tr\big[\mathcal{J}\big(s_-,u(s_-,\rho^{\mathbf{u}}_{s-})\big)(\rho^{\mathbf{u}}_{s-})\big]}-\rho^{\mathbf{u}}_{s-}\right]\mathbf{1}_{0<x<Tr[\mathcal{J}(s-,u(s-,\rho^{\mathbf{u}}_{s-}))(\rho^{\mathbf{u}}_{s-})]}N(ds,dx)\end{aligned}$$

where N is a Poisson Point Process on $\mathbb{R}^2$. As a consequence, if the process $(\rho^{\mathbf{u}}_t)$ exists, it gives rise to the process $(\tilde{N}_t)$ defined by

$$\tilde{N}_t=\int_0^t\int_{\mathbb{R}}\mathbf{1}_{0<x<Tr[\mathcal{J}(s-,u(s-,\rho^{\mathbf{u}}_{s-}))(\rho_{s-})]}N(ds,dx) \tag{42}$$

which is a counting process with stochastic intensity

$$t\to\int_0^t Tr[\mathcal{J}(s_-,u(s_-,\rho^{\mathbf{u}}_{s-}))(\rho^{\mathbf{u}}_{s-})]ds.$$

In [25], it is shown that the use of the Poisson process N allows to give a mathematical sense to equation (23) and allows to define properly the process $(\tilde{N}_t)$. Actually in (23), the problem concerns the definition and the existence of the process $(\tilde{N}_t)$. Indeed, the stochastic intensity of this process depends on the the solution of (23). In the way of writing the equation (23), in order to define $\tilde{N}_t$, it assumes implicitly the existence of the process solution whereas the equation is driven by $(\tilde{N}_t)$.

Now we consider the equation (41) as the jump-model of continuous time measurement with control. It will be justified later as limit of discrete quantum trajectories.

For the moment, we deal with the problem existence and uniqueness of a solution for this equation. Let us denote

$$\begin{aligned} R(t,a)(\rho) &= L(t,a)(\rho) + Tr[\mathcal{J}(t,a)(\rho)]\rho - \mathcal{J}(t,a)(\rho) \\ Q(t,a)(\rho) &= \left(\frac{\mathcal{J}(t,a)(\rho)}{Tr[\mathcal{J}(t,a)(\rho)]} - \rho\right)\mathbf{1}_{Tr[\mathcal{J}(t,a)(\rho)]>0} \end{aligned}$$

for all $t \geq 0$, for all $a \in \mathbb{R}$ and all state ρ. It was obvious that (41) is equivalent to

$$\begin{aligned} \rho_t^{\mathbf{u}} &= \rho_0 + \int_0^t R\big(s_-, u(s_-, \rho_{s-}^{\mathbf{u}})\big)(\rho_{s-}^{\mathbf{u}})ds \\ &\quad + \int_0^t \int_{\mathbb{R}} Q\big(s_-, u(s_-, \rho_{s-}^{\mathbf{u}})\big)(\rho_{s-}^{\mathbf{u}})\mathbf{1}_{0<x<Tr[\mathcal{J}(s-,u(s-,\rho_{s-}^{\mathbf{u}}))(\rho_{s-})]}N(ds,dx). \end{aligned}$$

In order to prove existence and uniqueness of a solution for such equations, a sufficient condition concerns the Lipschitz property for functions R an $\mathcal{J}$. In the same fashion of the diffusive case, this is not the case and a truncature method is used again. In the same way, we will have next to prove that the truncated solution takes values in the set of states. Regarding functions R an $\mathcal{J}$, the conditions for the Poisson case are expressed in the following remark.

Remark: As in the diffusive case, this remark concerns the regularity of the different functions. Firstly we suppose that R and $\mathcal{J}$ satisfy the local Lipschitz condition (25) defined in Section 2.1. Secondly as the set of states is compact, we can suppose for the stochastic intensity that for all $T > 0$ there exists a constant $K(T)$ such that

$$Tr[\mathcal{J}(t, u(t, X_t))(X_t)] \leq K(T)$$

for all $t \geq T$ and for all càdlàg process (X_t) with values in $\mathbb{M}_2(\mathbb{C})$. This previous condition implies the fact the stochastic intensity is bounded. Finally in order to consider the stochastic differential equation for all càdlàg process, we consider the function

$$\tilde{Q}(t,a)(\rho) = \left(\frac{\mathcal{J}(t,a)(\rho)}{Re\big(Tr[\mathcal{J}(t,a)(\rho)]\big)} - \rho\right)\mathbf{1}_{Re(Tr[\mathcal{J}(t,a)(\rho)])>0} \qquad (43)$$

and the stochastic differential equation

$$\rho_t^{\mathbf{u},k} = \rho_0 + \int_0^t R\big(s_-, u(s_-, \tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})\big)(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))ds \tag{44}$$

$$+ \int_0^t \int_{\mathbb{R}} \tilde{Q}\big(s_-, u(s_-, \tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})\big)(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))\mathbf{1}_{0<x<Re(Tr[\mathcal{J}(s-,u(s-,\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})))(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))]}N(ds, dx).$$

where $\tilde{\phi}^k$ is a truncature function defined in Section 3.1. As in the diffusive case, if a solution of the equation (44) takes value in the set of states, it is a solution of the equation (41). In addition to the diffusive case, we have to remark that if ρ is a state

$$Re(Tr[\mathcal{J}(t,a)(\rho)]) = Tr[\mathcal{J}(t,a)(\rho)] \geq 0$$

for all $t \geq 0$ and for all $a \in \mathbb{R}$.

Exactly in the same way as the diffusive case, if we show that a discrete quantum trajectory converges in distribution to a solution of the truncated equation (44), it involves that this solution takes values in the set of states. Let us first deal with the problem of existence and uniqueness of a solution for the equation (44). We have the following theorem due to Jacod and Protter in [16].

Theorem 7 *Let $(\Omega, \mathcal{F}, P)$ be a probability space of a Poisson point Process N. The stochastic differential equation*

$$\rho_t^{\mathbf{u},k} = \rho_0 + \int_0^t R\big(s_-, u(s_-, \tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})\big)(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))ds \tag{45}$$

$$+ \int_0^t \int_{\mathbb{R}} \tilde{Q}\big(s_-, u(s_-, \tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})\big)(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))\mathbf{1}_{0<x<Re(Tr[\mathcal{J}(s-,u(s-,\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})))(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))]}N(ds, dx)$$

admits a unique solution $\rho_t^{\mathbf{u},k}$ defined for al$t \geq 0$. Furthermore the process

$$\overline{N}_t = \int_0^t \int_{\mathbb{R}} \mathbf{1}_{0<x<Re(Tr[\mathcal{J}(s-,u(s-,\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})))(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))]}N(ds, dx)$$

allows to define the filtration $(\overline{\mathcal{F}}_t)$ where $\overline{\mathcal{F}}_t = \sigma\{\overline{N}_s, s \leq t\}$.

Hence the process

$$\overline{N}_t - \int_0^t \Big[Re(Tr[\mathcal{J}(s_-, u(s_-, \tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})))(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))\Big]^+ ds$$

is a $\overline{\mathcal{F}}_t$-martingale.

The term $(x)^+$ denotes $\max(0, x)$. Such theorem is treated in details in [25] for quantum trajectories without control. We give here a way to express the solution of (44) in a particular case.

Suppose that there exists a constant K such that:

$$\Big[Re(\mathcal{J}(t,u(t,X_t))(X_t))\Big]^+ < K, \tag{46}$$

for all $t \geq 0$ and all càdlàg process (X_t). With this property we can consider only the points of N contained in $\mathbb{R}\times[0,K]$. The random function

$$\mathcal{N}_t : t \to N(.,[0,t]\times[0,K])$$

defines then a standard Poisson process with intensity K. Let $T>0$, the Poisson Random Measure and the previous process generate on $[0,T]$ a sequence $\{(\tau_i,\xi_i), i\in\{1,\dots,\mathcal{N}_t)\}\}$. Each τ_i represents the jump time of the process $(\mathcal{N}_t)$. Moreover the random variables ξ_i are random uniform variables on $[0,K]$. Let $k>1$ be a fixed integer, we can write the solution of (44) in the following way:

$$\begin{aligned}
\rho_t^{\mathbf{u},k} &= \rho_0 + \int_0^t R(s_-,u(s_-,\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})))(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))ds \\
&\quad + \sum_{i=1}^{\mathcal{N}_t} Q(\tau_i-,u(\tau_i-,\tilde{\phi}^k(\rho_{\tau_i-}^{\mathbf{u},k}))(\tilde{\phi}^k(\rho_{\tau_i-}^{\mathbf{u},k}))\mathbf{1}_{0\leq\xi_i\leq(Re(Tr[\mathcal{J}(\tau_i-,u(\tau_i-,\tilde{\phi}^k(\rho_{\tau_i-}^{\mathbf{u},k})))(\tilde{\phi}^k(\rho_{\tau_i-}^{\mathbf{u},k})]))^+} \\
\overline{N}_t &= \sum_{i=1}^{\mathcal{N}_t}\mathbf{1}_{0\leq\xi_i\leq(Re(Tr[\mathcal{J}(\tau_i-,u(\tau_i-,\tilde{\phi}^k(\rho_{\tau_i-}^{\mathbf{u},k})))(\tilde{\phi}^k(\rho_{\tau_i-}^{\mathbf{u},k})]))^+}.
\end{aligned} \tag{47}$$

The general case is treated in details in [16]. Let us make more precise how the solution of (44) is defined from the expression (47) in the particular case (46). By applying Cauchy-Lipschitz Theorem, we consider the solution of the ordinary differential equation

$$\rho_t^{\mathbf{u},k} = \rho_0 + \int_0^t R(s_-,u(s_-,\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})))(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))ds. \tag{48}$$

It gives rise to the function

$$t \mapsto \Big[Re(Tr[\mathcal{J}(t,u(t,\tilde{\phi}^k(\rho_t^{\mathbf{u},k})))(\tilde{\phi}^k(\rho_t^{\mathbf{u},k}))\Big]^+.$$

Let define the first jump-time of the process $(\overline{N}_t)$. For this, we introduce the set

$$G_t = \{(x,y)\in\mathbb{R}^2/0<x\leq t, 0<y<\Big[Re(Tr[\mathcal{J}(x,u(x,\tilde{\phi}^k(\rho_x^{\mathbf{u},k})))(\tilde{\phi}^k(\rho_x^{\mathbf{u},k}))\Big]^+\},$$

and the random stopping time

$$T_1 = \inf\{t/N(G_t)=1\}.$$

As a consequence on $[0,T_1[$ the solution of (44) is given by the solution of the ordinary differential equation (48) and $\rho^{\mathbf{u},k}_{T_1}$ is defined by

$$\rho^{\mathbf{u},k}_{T_1}=\rho^{\mathbf{u},k}_{T_1-}+Q(T_1-,u(T_1-,\rho^{\mathbf{u},k}_{T_1-}))(\rho^{\mathbf{u},k}_{T_1-}).$$

We solve the ordinary differential equation after T_1 with this initial condition, and by a similar reasoning we shall define a second jump-time. Thus, we construct a sequence of jump-time (T_n). The boundness property (46) implies that the stochastic intensity is bounded. Hence, we can show $\lim T_n=\infty$ almost surely (see [16] or [25]).

The solution of (44) is then given by the solution of the ordinary differential equation

$$d\rho^{\mathbf{u},k}_t=R(t,u(t,\tilde{\phi}^k(\rho^{\mathbf{u},k}_t)))(\tilde{\phi}^k(\rho^{\mathbf{u},k}_t))dt$$

between the jump of the process $\overline{N}_t$. The process $\overline{N}_t$ corresponds to the number of point of the Poisson point process N included in the x axis and the curve

$$t\mapsto\left[Re(Tr[\mathcal{J}(t,u(t,\tilde{\phi}^k(\rho^{\mathbf{u},k}_t)))(\tilde{\phi}^k(\rho^{\mathbf{u},k}_t))\right]^+.$$

The general case is more technical but can be expressed in the same way.

In order to summarize the procedure to show the existence and uniqueness of solution, it is worth noticing that all the technical precaution are justified because we do not know that the equation preserves the property of being a state before showing the convergence result (it is the same problem in the diffusive case).

Now we investigate the convergence result. In a first time, the way to proceed is the same as in [25]. Next, we use another way to obtain the convergence result because we cannot applied the Theorem of Kurtz and Protter in this case.

From expression (40), define

$$\begin{aligned}
\rho^{\mathbf{u}}_n(t)&=&\rho^{\mathbf{u}}_{[nt]}\\
N_n(t)&=&\sum_{k=1}^{[nt]}\nu_k,\\
V_n(t)&=&\frac{[nt]}{n}\\
R_n(t,a)(\rho)&=&R([nt]/n,a)(\rho),\\
Q_n(t,a)(\rho)&=&Q([nt]/n,a)(\rho)\\
u_n(t,W)&=&u([nt]/n,W)
\end{aligned}$$

for all $t\geq 0$, for all $a\in\mathbb{R}$ and all $W\in\mathbb{M}_2(\mathbb{C})$. Hence the process $(\rho^{\mathbf{u}}_n(t))$ satisfies the stochastic differential equation

$$\begin{aligned}
\rho^{\mathbf{u}}_n(t)&=&\int_0^t\left[R_n\big(s_-,u_n(s_-,\rho^{\mathbf{u}}_n(s_-)\big)(\rho^{\mathbf{u}}_n(s_-))+\circ(1)\right]dV_n(s)\\
&&+\int_0^t\left[Q_n\big(s_-,u_n(s_-,\rho^{\mathbf{u}}_n(s_-))(\rho^{\mathbf{u}}_n(s_-)\big)+\circ(1)\right]dN_n(s).
\end{aligned}$$

In this case, we do not have directly an equivalent of the Donsker Theorem for the process $(N_n(t))$. Because of the stochastic intensity of $(\tilde{N}_t)$ which depends on the solution, it is actually not possible to prove the convergence of $(N_n(t))$ to $(\tilde{N}_t)$ independently of the convergence of $(\rho_{[nt]})$ to (ρ_t). Hence we cannot applied the result of Kurtz and Protter used in the diffusive case.

The convergence result is here obtained by using a random coupling method, that is, we realize the process $(\rho_{[nt]})$ in the probability space of the Poisson Point Process N. This method allows then to compare directly continuous and discrete quantum trajectories in the same probability space. It is described as follows.

Remember that the random variables $(\mathbf{1}_1^k)$ satisfy:

$$\begin{cases} \mathbf{1}_1^{k+1}(0) = 0 \quad \text{with probability} \quad p_{k+1}(n) = 1 - \frac{1}{n} Tr\big[\mathcal{J}(k/n, u(k/n, \rho_k^{\mathbf{u}}))(\rho_k^{\mathbf{u}})\big] + \circ\left(\frac{1}{n}\right) \\ \mathbf{1}_1^{k+1}(1) = 1 \quad \text{with probability} \quad q_{k+1}(n) = \frac{1}{n} Tr\big[\mathcal{J}(k/n, u(k/n, \rho_k^{\mathbf{u}}))(\rho_k^{\mathbf{u}})\big] + \circ\left(\frac{1}{n}\right) \end{cases}$$

We define the following sequence of random variable which are defined on the set of states

$$\tilde{\nu}_{k+1}(\eta, \omega) = \mathbf{1}_{N(\omega, G_k(\eta)) > 0} \tag{49}$$

where $G_k(\eta) = \big\{(t, u)/\frac{k}{n} \leq t < \frac{k+1}{n}, 0 \leq u \leq -n \ln(Tr[\mathcal{L}_0^{k+1}(n)(\eta)])\big\}$. Let $\rho_0 = \rho$ be any state and $T > 0$, we define the process $(\tilde{\rho}_k)$ for $k < [nT]$ by the recursive formula

$$\begin{aligned} \tilde{\rho}_{k+1}^{\mathbf{u}} &= \mathcal{L}_0^{k+1}(\tilde{\rho}_k^{\mathbf{u}}) + \mathcal{L}_1^{k+1}(\tilde{\rho}_k^{\mathbf{u}}) \\ &\quad + \left[-\frac{\mathcal{L}_0^{k+1}(\tilde{\rho}_k^{\mathbf{u}})}{Tr[\mathcal{L}_0^{k+1}(\tilde{\rho}_k^{\mathbf{u}})]} + \frac{\mathcal{L}_1^{k+1}(\tilde{\rho}_k^{\mathbf{u}})}{Tr[\mathcal{L}_1^{k+1}(\tilde{\rho}_k^{\mathbf{u}})]}\right] \big(\tilde{\nu}_{k+1}(\tilde{\rho}_k^{\mathbf{u}}, .) - Tr[\mathcal{L}_1^{k+1}(\tilde{\rho}_k^{\mathbf{u}})]\big). \end{aligned} \tag{50}$$

Thanks to properties of Poisson probability measure, the random variables $(\mathbf{1}_1^k)$ and $(\tilde{\nu}_k)$ have the same distribution. It involves the following property.

Proposition 3 *Let T be fixed. The discrete process $(\tilde{\rho}_k^{\mathbf{u}})_{k \leq [nT]}$ defined by (50) have the same distribution of the discrete quantum trajectory $(\rho_k^{\mathbf{u}})_{k \leq [nT]}$ defined by the quantum repeated measurement.*

The convergence result is then expressed as follows.

Theorem 8 *Let $T > 0$. Let $(\Omega, \mathcal{F}, P)$ be a probability space of a Poisson Point process N. Let $(\tilde{\rho}_{[nt]}^{\mathbf{u}})_{0 \leq t \leq T}$ be the discrete quantum trajectory defined by the recursive formula (50)*

Hence, for all $T > 0$ the process $(\tilde{\rho}_{[nt]}^{\mathbf{u}})_{0 \leq t \leq T}$ converges in distribution in $\mathcal{D}[0, T]$ (for the Skorohod topology) to the process $(\rho_t^{\mathbf{u}})$ solution of the stochastic differential equation:

$$\begin{aligned} \rho_t^{\mathbf{u}} &= \rho_0 + \int_0^t R(s_-, u(s_-, \rho_{s_-}^{\mathbf{u}})(\rho_{s_-}^{\mathbf{u}}) ds \\ &\quad + \int_0^t \int_{\mathbb{R}} Q(s_-, u(s_-, \rho_{s_-}^{\mathbf{u}})(\rho_{s_-}^{\mathbf{u}}) \mathbf{1}_{0 < x < Tr[\mathcal{J}(s_-, u(s_-, \rho_{s_-}^{\mathbf{u}}))(\rho_{s_-}^{\mathbf{u}})])} N(., ds, dx). \end{aligned} \tag{51}$$

This theorem relies on the fact that the process $(\tilde{\rho}^{\mathbf{u}}_{[nt]})$ satisfies the same asymptotic of the discrete quantum trajectory $(\rho^{\mathbf{u}}_{[nt]})$; we have namely

$$\begin{aligned}\tilde{\rho}_{[nt]} &= \rho_0 + \sum_{k=0}^{[nt]-1} \frac{1}{n}\left[R(k/n, u(k/n, \tilde{\rho}^{\mathbf{u}}_k))(\tilde{\rho}^{\mathbf{u}}_k) + \circ(1)\right] \\ &\quad + \sum_{k=0}^{[nt]-1} \left[Q(k/n, u(k/n, \tilde{\rho}^{\mathbf{u}}_k))(\tilde{\rho}^{\mathbf{u}}_k) + \circ(1)\right] \tilde{\nu}_{k+1}(\rho^{\mathbf{u}}_k, .). \end{aligned} \tag{52}$$

The complete proof of this theorem is very technical. The idea is to compare the discrete process $(\rho_{[nt]})$ with an Euler Scheme of the solution of the jump-equation. More details for such techniques can be found in [25] where the case without control is entirely developed.

In the next section, we expose examples and applications of such stochastic models.

3 Examples and Applications

This section is devoted to some applications of quantum measurement with control. In a first time, by a discrete model, we justify a stochastic model for the experience of Resonance fluorescence. The setup is the one of a laser driving an atom in presence of a photon counter. In a second time, we present general results in Stochastic Control theory applied to quantum trajectories.

3.1 Laser Monitoring Atom: Resonance Fluorescence

We here describe a discrete model of an atom monitored by a laser. A measurement is performed by a photon counter which detects the photon emission. The setup of repeated quantum interactions is described as follows.

The length time of interaction is chosen to be $h = 1/n$. Let us describe one interaction. Here we need three basis spaces. The atom system is represented by $\mathcal{H}_0$ equipped with a state ρ. The laser is representing by $(\mathcal{H}^l, \mu^l)$ and the photon counter by $(\mathcal{H}^c, \beta^c)$. Each Hilbert space are $\mathbb{C}^2$ endowed with the orthonormal basis (Ω, X) and the unitary operator is denoted by U. The compound system after interaction is:

$$\mathcal{H}_0 \otimes \mathcal{H}^l \otimes \mathcal{H}^c,$$

and the state after interaction is:

$$\alpha = U(\rho \otimes \mu^l \otimes \beta^c)U^\star.$$

The appropriate orthonormal basis $\mathcal{H}_0 \otimes \mathcal{H}^l \otimes \mathcal{H}^c$, in this case, is $\Omega\otimes\Omega\otimes\Omega$, $X\otimes\Omega\otimes\Omega$, $\Omega\otimes X\otimes\Omega$, $X\otimes X\otimes\Omega, \Omega\otimes\Omega\otimes X$, $X\otimes\Omega\otimes X$, $\Omega\otimes X\otimes X$, $X\otimes X\otimes X$. As in the presentation of the discrete two level atom in contact with a spin chain, the unitary operator is here considered as a 4×4 matrix $U = (L_{i,j}(n))_{0 \le i,j \le 3}$ where each $L_{ij}(n)$ are operator on $\mathcal{H}_0$.

If the different state of the laser and the counter as of the form

$$\mu^l = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \beta^c = |\Omega\rangle\langle\Omega|,$$

for the state $\alpha = (\alpha_{uv})_{0 \le u,v \le 3}$ after interaction, we have

$$\alpha_{uv} = \Big(aL_{u0}(n)\rho + bL_{u1}(n)\rho\Big)L^\star_{v0}(n) + \Big(cL_{u0}(n)\rho + dL_{u1}(n)\rho\Big)L^\star_{v1}(n). \tag{53}$$

The measurement is performed on the counter photon side. Let A denotes any observable of $\mathcal{H}^c$ then $I \otimes I \otimes A$ denotes the corresponding observable on $\mathcal{H}_0 \otimes \mathcal{H}^l \otimes \mathcal{H}^c$. We perform a measurement and by partial trace operation with respect to $\mathcal{H}^l \otimes \mathcal{H}^c$ we obtain a new state on $\mathcal{H}_0$.

The control is rendered by the modification at each interaction of the intensity of the laser. This modification is here taken into account by the reference state of the laser. The reference state at the k-th interaction is denoted by μ^l_k. In the continuous case of Resonance fluorescence, the state of a laser is usually described by a coherent vector on a Fock space (see [9]). From works of Attal and Pautrat in approximation of Fock space ([1],[24]), in our context the discrete state of the laser can be described by

$$\mu^l_k = \begin{pmatrix} a(k/n) & b(k/n) \\ c(k/n) & d(k/n) \end{pmatrix} = \frac{1}{1+|h(k/n)|^2}\begin{pmatrix} 1 & h(k/n) \\ \overline{h}(k/n) & |h(k/n)|^2 \end{pmatrix}. \tag{54}$$

The function h represents the evolution of the intensity of the laser and depends naturally on n.

If ρ_k denotes the state on $\mathcal{H}_0$ after k first measurement, the state

$$\alpha^{k+1}(n) = \big(\alpha^{k+1}_{uv}(n)\big)_{0 \le u,v \le 3} = U_{k+1}(n)(\rho_k \otimes \mu^l_k \otimes \beta^c)U^\star_{k+1}(n)$$

after interaction satisfies

$$\begin{aligned} \alpha^{k+1}_{uv}(n) &= \Big(a(k/n)L_{u0}(n)\rho_k + b(k/n)L_{u1}(n)\rho_k\Big)L^\star_{v0}(n) \\ &\quad + \Big(c(k/n)L_{u0}(n)\rho_k + d(k/n)L_{u1}(n)\rho_k\Big)L^\star_{v1}(n) \end{aligned}$$

Remark: Let us stress that is not directly the framework of Section 1. Here, the control is namely not rendered by the modification of the unitary evolution. Moreover the interacting system is described by $(\mathcal{H}^l \otimes \mathcal{H}^c, \mu^l_k \otimes \beta)$ and $\mu^l_k \otimes \beta$ is not of the form $|X_0\rangle\langle X_0|$ as in Section 1. In order to translate this setting in the case of discrete models of Section 1, one can use the G.N.S Representation theory of a finite dimensional Hilbert space ([19],[18]). This theory allows to consider the state $\mu^l_k \otimes \beta$ as a state of the form $|X_0\rangle\langle X_0|$ in a biggest Hilbert space. The G.N.S representation modifies then the expression of operator U_k, and the control expressed in $\mu^l_k \otimes \beta$ is again expressed in the new expression of U_k (see [2] for more details). In our case, we do not use such theory because it is more explicit to make directly computations to reach the discrete equation in asymptotic form.

Let us present the result. The principle of measurement is the same as in Section 1. The counting case is also given by a diagonal observable of $\mathcal{H}_c$. We shall focus on this case which renders the emission of photon ([9]). The asymptotic for the unitary operator follows the asymptotic of Attal-Pautrat in [4]. Let $\delta_{ij} = 1$ if $i = j$ we denote:

$$\epsilon_{ij} = \frac{1}{2}(\delta_{0i} + \delta_{0j})$$

The coefficients must follow the convergence condition:

$$\lim_{n\to\infty} n^{\epsilon_{ij}}(L_{ij}(n) - \delta_{ij}I) = L_{ij}$$

where L_{ij} are operator on $\mathcal{H}_0$.

Let $P_0 = |\Omega\rangle\langle\Omega|$ and $P_1 = |X\rangle\langle X|$ be eigenprojectors of a diagonal observable A. If ρ_k denotes the random state after k measurements we denote:

$$\begin{aligned}
\mathcal{L}_0^{k+1}(\rho_k) &= \mathbf{E}_0[I \otimes I \otimes P_0(U_{k+1}(n)(\rho_k \otimes \mu_k^l \otimes \beta)U_{k+1}^\star(n))I \otimes I \otimes P_0] \\
&= \alpha_{00}^{k+1}(n) + \alpha_{11}^{k+1}(n) \\
\mathcal{L}_1^{k+1}(\rho_k) &= \mathbf{E}_0[I \otimes I \otimes P_1(U_{k+1}(n)(\rho_k \otimes \mu_k^l \otimes \beta)U_{k+1}^\star(n))I \otimes I \otimes P_1] \\
&= \alpha_{22}^{k+1}(n) + \alpha_{33}^{k+1}(n) \qquad (55)
\end{aligned}$$

This is namely the two non normalized state, the operator $\mathcal{L}_0^{k+1}(\rho_k)$ appears with probability $p_{k+1} = Tr[\mathcal{L}_0^{k+1}(\rho_k)]$ and $\mathcal{L}_1^{k+1}(\rho_k)$ with probability $q_{k+1} = Tr[\mathcal{L}_0^{k+1}(\rho_k)]$.

From works of Attal-Pautrat in approximation and asymptotic in Fock space, we put

$$h(k/n) = \frac{1}{\sqrt{n}} f(k/n) + \circ\left(\frac{1}{n}\right),$$

where f is a function from $\mathbb{R}$ to $\mathbb{C}$. In the same way of Section 2, we assume that the intensity of the laser f is continuous.

With the same arguments of Section 1, the evolution of the discrete quantum trajectory is described by

$$\rho_k = \frac{\mathcal{L}_0^{k+1}(\rho_k)}{p_{k+1}} + \left[-\frac{\mathcal{L}_0^{k+1}(\rho_k)}{p_{k+1}} + \frac{\mathcal{L}_1^{k+1}(\rho_k)}{q_{k+1}}\right] \mathbf{1}_1^{k+1} \qquad (56)$$

For a further use, convergence results will be established in the case $L_{01} = -L_{10}^\star$, and $L_{11} = L_{21} = L_{31} = L_{30} = 0$. Conditions about asymptotic of U and the fact that it is a unitary-operator imply that

$$L_{00} = -(iH + \frac{1}{2}\sum_{i=1}^{2} L_{i0}^\star L_{i0}) \qquad (57)$$

In the same way of Section 2.2, convergence result in this situation is expressed as follows.

Proposition 4 *Let* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *be a probability space of a Poisson point process* N *on* $\mathbb{R}^2$.

The discrete quantum trajectory $(\rho_{[nt]})_{0\leq t\leq T}$ *defined by the equation* (56) *weakly converges in* $\mathcal{D}([0,T])$ *for all* T *to the solution of the following stochastic differential equation:*

$$\begin{aligned}\rho_t &= \rho_0 + \int_0^t \Bigg[-i[H,\rho_{s-}] + \frac{1}{2}\{\sum_{i=1}^2 L_{i0}^\star L_{i0}, \rho_{s-}\} + L_{10}\rho_{s-}L_{10}^\star \\ &\qquad +[\overline{f}(s_-)L_{10}\rho_{s-} - f(s_-)L_{10}^\star, \rho_{s-}] - Tr[L_{20}\rho_{s-}L_{20}^\star]\rho_{s-}\Bigg]ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \left[-\rho_{s-} + \frac{L_{20}\rho_{s-}L_{20}^\star}{Tr[L_{20}\rho_{s-}L_{20}^\star]}\right] \mathbf{1}_{0<x<Tr[L_{20}\rho_{s-}L_{20}^\star]} N(dx,ds). \end{aligned} \tag{58}$$

Proof: For example we have the following asymptotic for $\mathcal{L}_0^{k+1}(\rho_k)$:

$$\begin{aligned}&\mathcal{L}_0(\rho_k) = \\ &\rho_k + \frac{1}{n}\left[L_{00}\rho + \rho L_{00}^\star + L_{10}\rho L_{10}^\star + f(\frac{k}{n})[L_{01}\rho + \rho L_{10}^\star] + \overline{f}(\frac{k}{n})[L_{10}\rho + \rho L_{01}^\star]\right] + \circ\left(\frac{1}{n}\right)\end{aligned}$$

This above asymptotic, the condition about the operator L_{ij} and the theorem (8) prove the proposition. □

The stochastic differential equation (58) is then the continuous time stochastic model of Resonance fluorescence. In this model, the control is deterministic. Before to give an application of stochastic control , let us briefly expose a use of the laser monitoring atom model.

Consider the special case, where the Hamiltonian $H = 0$. Let put

$$C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ L_{10} = k_l C,\ L_{20} = k_c C,$$

with $|k_l|^2 + |k_c|^2 = 1$. The constant k_f and k_c are called decay rates ([9]).

Without control, the stochastic model of a two level atom in presence of a photon counter ([25]) is given by:

$$\begin{aligned}\mu_t &= \mu_0 + \int_0^t \left[+ \frac{1}{2}\{C,\mu_{s-}\} + C\mu_{s-}C^\star - Tr[C\mu_{s-}C^\star]\mu_{s-}\right]ds \\ &\quad + \int_0^t\int_{\mathbb{R}} \left[-\mu_{s-} + \frac{C\mu_{s-}C^\star}{Tr[C\mu_{s-}C^\star]}\right] \mathbf{1}_{0<x<Tr[C\mu_{s-}C^\star]} N(dx,ds).\end{aligned} \tag{59}$$

Let denote $\tilde{N}_t = \int_0^t \int_{\mathbb{R}} \mathbf{1}_{0<x<Tr[C\mu_{s-}C^\star]} N(dx,ds)$ and $T = \inf\{t>0;\ \tilde{N}_t > 0\}$. In [5] it was proved that:

$$\mu_t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |\Omega\rangle\langle\Omega|. \tag{60}$$

for all $t > T$. Physically, it means that at most one photon appears on the photon counter. The mathematical reason is that if we write the equation (59) in the following way:

$$\mu_t = \int_0^t \Psi(\mu_{s-})ds + \int_0^t \Phi(\mu_{s-})d\tilde{N}_s,$$

we have for $\mu = |\Omega\rangle\langle\Omega|$

$$\Phi(\mu) = \Psi(\mu) = 0.$$

The state $|\Omega\rangle\langle\Omega|$ is an equilibrium state.

In the presence of laser, the control f gives rise to the term $[\overline{f}L_{10} - fL_{10}^\star, .] = [k_l\overline{f}C - \overline{k_l}fC^\star, .]$. Hence if $\mu = |\Omega\rangle\langle\Omega|$, we still have $\Phi(\mu) = 0$ but we do not have anymore $\Psi(\mu) = 0$ and the property (60) is not satisfied. The state $|\Omega\rangle\langle\Omega|$ is no more an equilibrium state. As a consequence it is possible to observe more than one photon in the photon counter.

In the next section we deal with general strategy and the particular problem of optimal control. Considerations about optimal control is an interesting mean to point out the importance of Markovian strategy.

3.2 Optimal Control

This section is then devoted to what is called the "optimal control" problem. It deals with finding a particular control strategy which must satisfy optimization constraint. In this section, we give the classical mathematical description of such problem and investigate general results in the discrete and in the continuous model of controlled quantum trajectories. Let us begin with the discrete model.

3.2.1 The Discrete Case

We come back to the description of a discrete quantum trajectory for a two-level system as a Markov chain.

Let n be fixed, thanks to Theorem 2, a discrete controlled quantum trajectory $(\rho_k^{\mathbf{u}})$ is described as follows. Let ρ be any state, if $\rho_k^{\mathbf{u}} = \rho$ then $\rho_{k+1}^{\mathbf{u}}$ takes one of the values:

$$\mathcal{H}_i^{\mathbf{u},k}(\rho) = \frac{L_{i0}(k/n, u_k(n))(\rho)L_{i0}^\star(k/n, u_k(n))}{Tr[L_{i0}(k/n, u_k(n))(\rho)L_{i0}^\star(k/n, u_k(n))]} \; i = 0, 1$$

with probability,

$$\begin{array}{rcl} p_{k+1}^{\mathbf{u}}(\rho) & = & Tr[L_{00}(k/n, u_k(n))(\rho)L_{00}^\star(k/n, u_k(n))] \text{ for } i = 0 \\ q_{k+1}^{\mathbf{u}}(\rho) & = & Tr[L_{10}(k/n, u_k(n))(\rho)L_{10}^\star(k/n, u_k(n))] \text{ for } i = 1. \end{array}$$

With this previous description, the property of a strategy (u_k) can be enlarged. We can namely consider more general strategies such that for all k the term u_k depend on all (ρ_i) for $i \leq k$. We define $\mathcal{U}$ the set of all admissible strategies which satisfy this condition.

Let us stress that in this situation, the discrete quantum trajectory is no more a Markov chain because the strategy at time k depends on all the past of the strategy.

With this remark concerning the definition of strategies, we can expose the general problem of "optimal control". In this article, we only consider finite horizon problem. It is described as follows.

Let N be a fixed integer and let c and ϕ be two measurable function, the optimal control problem in finite horizon is to consider what is called the "optimal cost":

$$\min_{\mathbf{u}\in\mathcal{U}} \mathbf{E}\left[\sum_{k=0}^{N-1} c(k,\rho_k^{\mathbf{u}},u_k)+\phi(\rho_N^{\mathbf{u}})\right]. \tag{61}$$

If there is some strategy which realizes the minimum, this strategy is called the "optimal strategy". Let us investigate the classical result in stochastic control for the optimal strategy in this case.

For this we define:

$$V^k(\rho)=\min_{\mathbf{u}\in\mathcal{U}} \mathbf{E}\left[\sum_{j=k}^{N-1} c(k,\rho_j^{\mathbf{u}},u_j)+\phi(\rho_N^{\mathbf{u}})\Big/\rho_n^{\mathbf{u}}=\rho\right].$$

Remark The function c and ϕ are determined by the optimization constraint imposed by the experience. The equation which appears in the following theorem is called the cost equation and the function c and ϕ are called cost function.

Theorem 9 *Let $\mathcal{U}$ be a compact set and suppose that c is a continuous function. The solution of:*

$$\left\{\begin{array}{lcl} V^k(\rho) & = & \min_{u\in\mathcal{U}}\{p_{k+1}^{\mathbf{u}}(\rho)\mathcal{H}_0^{\mathbf{u},k}(\rho)+q_{k+1}^{\mathbf{u}}(\rho)\mathcal{H}_1^{\mathbf{u},k}(\rho)+c(k,\rho,u_k)\} \\ V^N(\rho) & = & \phi(\rho) \end{array}\right. \tag{62}$$

give the optimal cost:

$$V^k(\rho)=\min_{\mathbf{u}\in\mathcal{U}} \mathbf{E}\left[\sum_{j=k}^{N-1} c(k,\rho_j^{\mathbf{u}},u_j)+\phi(\rho_N^{\mathbf{u}})\Big/\rho_n=\rho\right].$$

The optimal strategy is given by:

$$u^\star:\rho\to u_k^\star(\rho)\in\arg\min_{\mathbf{u}\in\mathcal{U}}\{p_{k+1}^{\mathbf{u}}(\rho)\mathcal{H}_0^{\mathbf{u},k}(\rho)+q_{k+1}^{\mathbf{u}}(\rho)\mathcal{H}_1^{\mathbf{u},k}(\rho)+c(k,\rho,u_k)\}. \tag{63}$$

Furthermore this strategy is Markovian.

Proof: The proof is based of what is called dynamic programming in stochastic control theory. Let $\mathbf{u}$ be any strategy and let V defined by the formula (62), we have

$$\begin{array}{rl} & \mathbf{E}[(V^{k+1}\left(\rho_{k+1}^{\mathbf{u}}\right)-V^k\left(\rho_k^{\mathbf{u}}\right))/\sigma\{\rho_i^{\mathbf{u}},i\leq k\}] \\ = & p_{k+1}^{\mathbf{u}}V^{k+1}\left(\mathcal{H}_0^{\mathbf{u},k}(\rho_k^{\mathbf{u}})\right)+q_{k+1}^{\mathbf{u}}V^{k+1}\left(\mathcal{H}_1^{\mathbf{u},k}(\rho_k^{\mathbf{u}})\right)-V^k(\rho_k^{\mathbf{u}})) \end{array}$$

then we have

$$
\begin{aligned}
& \mathbf{E}\left[V^N(\rho_N^{\mathbf{u}}) - V^0(\rho)\right] \\
= & \sum_{k=0}^{N-1} \mathbf{E}\left[V^{k+1}(\rho_{k+1}^{\mathbf{u}}) - V^k(\rho_k^{\mathbf{u}})\right] \\
= & \sum_{k=0}^{N-1} \mathbf{E}\left[p_{k+1}^{\mathbf{u}} V^{k+1}\left(\mathcal{H}_0^{\mathbf{u},k}(\rho_k^{\mathbf{u}})\right) + q_{k+1}^{\mathbf{u}} V^{k+1}\left(\mathcal{H}_1^{\mathbf{u},k}(\rho_k^{\mathbf{u}})\right) - V^k(\rho_k^{\mathbf{u}}))\right] \\
\geq & -\sum_{k=0}^{N-1} \mathbf{E}\left[c(k, \rho_k^{\mathbf{u}}, u_k)\right] \ \text{(by definition of the min).}
\end{aligned}
$$

Hence for all strategy $\mathbf{u}$, we have

$$
V^0(\rho) \leq \mathbf{E}\left[\sum_{k=0}^{N-1} c(k, \rho_k^{\mathbf{u}}, u_k) + \phi(\rho_N^{\mathbf{u}})\right].
$$

Moreover we have equality if we choose the strategy given by the formula (63). This strategy is Markovian because the function c depends only on ρ_k at time k. □

The system (62) which describes the cost equation is called the discrete Hamilton-Jacobi Bellman equation.

The fact that the optimal strategy is Markovian is another justification of the choice of such model of control for the discrete quantum trajectory. This theorem claims that we need just Markovian strategy in order to solve the "optimal control" problem.

The next last section is devoted to the same investigation in continuous time models of quantum trajectories.

3.2.2 The Continuous Case

In the third section, we have proved the Poisson and the diffusion approximation in quantum measurement theory. We have the diffusive evolution equation

$$
\rho_t = \rho_0 + \int_0^t L(s, \rho_s^{\mathbf{u}}, u(s, \rho_s^{\mathbf{u}}))ds + \int_0^t \Theta(s, \rho_s^{\mathbf{u}}, u(s, \rho_s^{\mathbf{u}}))dW_s, \tag{64}
$$

and the jump-equation is

$$
\begin{aligned}
\rho_t = & \rho_0 + \int_0^t R(s_-, \rho_{s-}^{\mathbf{u}}, u(s_-, \rho_{s-}^{\mathbf{u}}))ds & (65) \\
& + \int_0^t \int_{\mathbb{R}} Q(s, \rho_{s-}^{\mathbf{u}}, u(s_-, \rho_{s-}^{\mathbf{u}}))\mathbf{1}_{0<x<Tr[\mathcal{J}(s-,u(s-,\rho_{s-}^{\mathbf{u}}))(\rho_{s-}^{\mathbf{u}})]} N(dx, ds), & (66)
\end{aligned}
$$

where the functions L, Θ, R and Q are defined in Section 2.

In this section, we consider the same problem of "optimal control" as in the discrete case. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space where we consider the diffusive equation

$$\rho_t = \rho_0 + \int_0^t L(s, \rho_s^{\mathbf{u}}, u_s)ds + \int_0^t \Theta(s, \rho_s^{\mathbf{u}}, u_s)dW_s,$$

and the jump-equation

$$\begin{aligned} \rho_t &= \rho_0 + \int_0^t R(s-, \rho_{s-}^{\mathbf{u}}, u_{s-})ds \\ &\quad + \int_0^t \int_{\mathbb{R}} Q(s, \rho_{s-}^{\mathbf{u}}, u_{s-})\mathbf{1}_{0<x<Tr[\mathcal{J}(s-,u_{s-})(\rho_{s-}^{\mathbf{u}})]}N(dx, ds) \end{aligned}$$

where the strategy $\mathbf{u} = (u_t)$ is just supposed to be a function $\mathcal{F}_t$ adapted (not only Markovian). In the case where $\mathcal{F}_t$ corresponds to the filtration generated by the process (ρ_t), we recover the same definition as the discrete case. Concerning existence and uniqueness of a solution, with the condition (25) of Section 2.1 for the functions L, R and θ the previous equations admit a unique solution. Furthermore the solution takes values in the set of states on $\mathcal{H}_0$. The set of all admissible strategy which satisfy the condition of adaptation is also denoted by $\mathcal{U}$. The optimal control problem in this situation is expressed as follows.

Let c and ϕ be two cost function. Let $T > 0$, the optimal problem in finite horizon is given by

$$\min_{\mathbf{u}\in\mathcal{U}} \mathbf{E}\left[\int_0^T c(s, \rho_s^{\mathbf{u}}, u_s)ds + \phi(\rho_T^{\mathbf{u}})\right]. \tag{67}$$

As in the discrete model, we introduce the following function:

$$V(t, \rho) = \min_{\mathbf{u}\in\mathcal{U}} \mathbf{E}\left[\int_t^T c(s, \rho_s^{\mathbf{u}}, u_s)ds + \phi(\rho_T^{\mathbf{u}}) \Big/ \rho_t^{\mathbf{u}} = \rho\right], \tag{68}$$

which satisfies

$$V(0, \rho_0) = \min_{\mathbf{u}\in\mathcal{U}} \mathbf{E}\left[\int_0^T c(s, \rho_s^{\mathbf{u}}, u_s)ds + \phi(\rho_T^{\mathbf{u}})\right].$$

The function (68) represents the result of optimal control after t assuming $\rho_t = \rho$.

In this article, we just give the result for the optimal control problem for the diffusive case. A similar result for the Poisson case can be found in [10].

As in the discrete case, it appears a continuous time version of the Hamilton-Jacobi-Bellmann Equation. The usual expression of this equation use the notion of infinitesimal generator of $(\rho_t^{\mathbf{u}})$. It is described as follows in our context. A quantum trajectory $(\rho_t^{\mathbf{u}})$ is considered as a process which takes values in $\mathbb{R}^3$ with the identification of the state and the Bloch sphere $\mathbb{B}_1(\mathbb{R}^3) = \{(x, y, z) \in \mathbb{R}^3 / x^2 + y^2 + z^2 \leq 1\}$, that is,

$$\begin{array}{rccc} \Phi: & \mathbb{B}_1(\mathbb{R}^3) & \longmapsto & \mathbb{M}_2(\mathbb{C}) \\ & (x, y, z) & \longrightarrow & \frac{1}{2}\begin{pmatrix} 1+x & y+iz \\ y-iz & 1-x \end{pmatrix} \end{array}$$

The map Φ is injective and its range is the set of states. By considering that the functions L and Θ are applications from $\mathbb{R}_+ \times \mathbb{R}^3$ to $\mathbb{R}^3$, the stochastic differential equation concerning the diffusive case can be written as a system of stochastic differential equation on $\mathbb{R}^3$ of the form:

$$(\rho_t^{\mathbf{u}})_i = \rho_0 + \int_0^t L_i(s, \rho_s^{\mathbf{u}}, u_s)ds + \int_0^t \Theta_i(s, \rho_s^{\mathbf{u}}, u_s)dW_s \quad i \in \{1, 2, 3\}$$

where $(\rho_t^{\mathbf{u}})_i$ (respectively Θ_i and L_i) corresponds to the coordinate function of $\rho_t^{\mathbf{u}}$ (respectively Θ and L).

We introduce the 3×3 matrix Π defined by $\Pi_{ij} = \Theta_i \Theta_j$. The infinitesimal generator $\mathcal{A}^{u,t}$ of the process $(\rho_t^{\mathbf{u}})$ acts on the functions f which are C^2 and bounded in the following way

$$\mathcal{A}^{u,t} f(x) = \frac{1}{2} \sum_{i,j=1}^{3} \Pi_{ij}(t, x, u) \frac{\partial f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{3} L_i(t, x, u) \frac{\partial f(x)}{\partial x_i}. \tag{69}$$

for all $t \geq 0$, $u \in \mathbb{R}$ and $x \in \mathbb{R}^3$. In particular if u is a fixed constant, let (ρ_t) be the solution of

$$(\rho_t)_i = \rho_0 + \int_0^t L_i(s, \rho_s, u)ds + \int_0^t \Theta_i(s, \rho_s, u)dW_s \quad i \in \{1, 2, 3\}.$$

Hence for all function f which is C^2 and bounded, the following process

$$\mathcal{M}_t = f(\rho_t) - f(\rho_0) - \int_0^t \mathcal{A}^{u,s} f(\rho_s)ds$$

is a martingale for the filtration generated by (ρ_t).

The following theorem express the result in optimal control for the diffusive quantum trajectory.

Theorem 10 *Suppose there is a function $(t, \rho) \to V(t, \rho)$ which is C^1 in t and C^2 in ρ such that:*

$$\left\{ \begin{array}{lcl} \frac{\partial V(t,\rho)}{\partial t} + \min_{u \in \mathcal{U}} \{\mathcal{A}^{u,t} V(t, \rho) + c(t, \rho, u)\} & = & 0 \\ V(T, \rho) & = & \phi(\rho) \end{array} \right. \tag{70}$$

where $\mathcal{A}^{u,t} f(x)$ is defined by the expression (69). The function V gives the solution of the optimal problem, that is,

$$V(t, \rho) = \min_{\mathbf{u} \in \mathcal{U}} \mathbf{E} \left[\int_t^T c(s, \rho_s^{\mathbf{u}}, u_s)ds + \phi(\rho_T^{\mathbf{u}}) \Big/ \rho_t^{\mathbf{u}} = \rho \right].$$

Furthermore if the strategy $\mathbf{u}$ defined by

$$u^\star(t, \rho) \in \arg\min_{u \in \mathcal{U}} \{\mathcal{A}^{u,t} V(t, \rho) + c(t, \rho, u)\} \tag{71}$$

is an admissible strategy then it defines an optimal strategy. Moreover this strategy is Markovian.

The equation (70) is the Hamilton-Jacobi-Bellmann equation in the continuous case.

A proof of this theorem can be found in [22] or [28]. The interest of such theorem in our context is to notice that the optimal strategy is Markovian, this confirms the choice of such strategy in the model of quantum trajectories with control.

A similar result holds for the Poisson case. The infinitesimal generator for such process is given in [13], explicit computations can be found in [27].

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